If you have a conflict with the midterm, you should have gotten information via email from me about the alternate sitting.

If you have a conflict and have not gotten this email, email me ASAP:

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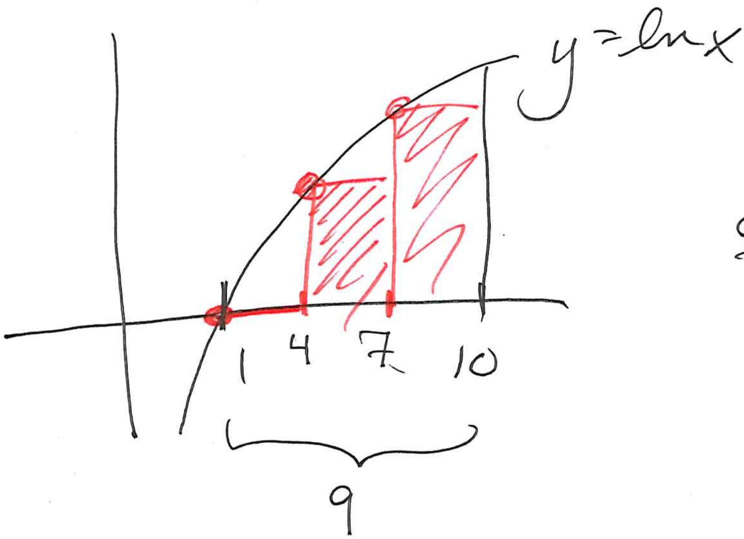
Riemann Sums

Approx area under $y = f(x)$ by adding rectangles

(ex) Approx area under $y = \ln x$, over $[1, 10]$ using $n = 3$ subintervals

3 types of RS: Left, Right, Midpoint



Base of □: $\Delta x = \frac{10-1}{3} = 3$

Left RS, height of □s:

x_k^*: 1, 4, 7

height: $\ln(1) = 0$
$\ln(4)$
$\ln(7)$

Left RS:
$3(0) + 3\ln 4 + 3\ln 7$
$= 3(\ln 4 + \ln 7)$
$= \boxed{3\ln(28)}$

More efficient notation: Σ notation

"sigma"
"S" like "sum"

$$\sum_{k=a}^{b} f(k)$$

a, b: bounds (integers)
k: index (take all integers, $k=a$ to $k=b$)
$f(k)$: summands

ex $$\sum_{k=2}^{4} (2k+5) = \underset{(k=2)}{(4+5)} + \underset{(k=3)}{(6+5)} + \underset{(k=4)}{(8+5)}$$

$$= 18 + 15 = \boxed{33}$$

ex Evaluate:

$$\sum_{k=5}^{8}(k^2-k) = \underset{(k=5)}{(25-5)} + \underset{(k=6)}{(36-6)} + \underset{(k=7)}{(49-7)} + \underset{(k=8)}{(64-8)}$$

$$\sum_{k=5}^{7} 3k = \underset{(k=5)}{3\cdot 5} + \underset{(k=6)}{3\cdot 6} + \underset{(k=7)}{3\cdot 7} = 3(5+6+7) = 3\sum_{k=5}^{7} k$$

$$\sum_{k=-5}^{-2} 8 = \underset{(k=-5)}{8} + \underset{(k=-4)}{8} + \underset{(k=-3)}{8} + \underset{(k=-2)}{8} = 4\cdot 8$$

$$\sum_{k=1}^{n} 17 = 17n \quad \Big\| \quad \sum_{k=a}^{b} c = c(b-a+1)$$

ex OK or NOT OK

A. $\sum_{k=1}^{15} (k^2 - k) = \left(\sum_{k=1}^{15} k^2 \right) - \left(\sum_{k=1}^{15} k \right)$

B. $\sum_{k=1}^{15} k(k-1) = k \sum_{k=1}^{15} (k-1)$

C. $\sum_{k=1}^{15} (k-1) = -15 + \sum_{k=1}^{15} k$

A: commutativity (order doesn't matter)

$$\sum_{k=1}^{15} (k^2-k) = 1^2-1 + 2^2-2 + 3^2-3 \ldots$$

$$= 1^2+2^2+3^2+\ldots -1-2-3\ldots$$

$$= 1^2+2^2+3^2+\ldots -(1+2+3\ldots)$$

$$= \sum_{k=1}^{15} k^2 - \sum_{k=1}^{15} k$$

B: NOT ok

$$\underbrace{\sum_{k=1}^{15} k(k-1)}_{\text{a number}} = 1(0) + 2(1) + 3(2) + 4(3) + \cdots + 15(14)$$

changing

can't factor

Compare: $\overset{???}{k} \underbrace{\sum_{k=1}^{15} (k-1)}_{\text{number}}$

C: Same as A

$$\sum_{k=1}^{15}(k-1) = 1-1 + 2-1 + 3-1 + 4-1 + \dots + 15-1$$

$$= \underbrace{(-1)+(-1)+(-1)+\dots+(-1)}_{\text{Fifteen times}}$$

$$+ 1+2+3+4+\dots+15$$

$$= 15(-1) + \sum_{k=1}^{15} k$$

$$= -15 + \sum_{k=1}^{15} k$$

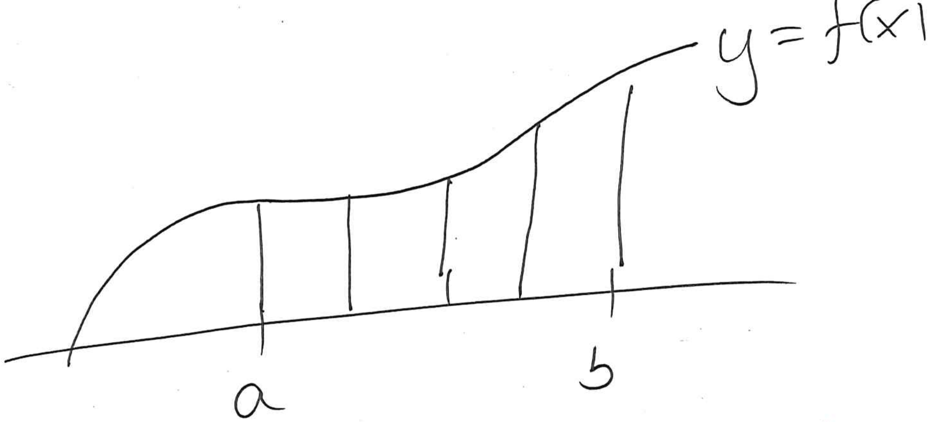
Equivalently:

$$\sum_{k=1}^{15}(k-1) = \sum_{k=1}^{15} k + \sum_{k=1}^{15}(-1) = \sum_{k=1}^{15} k + (-15)$$

We'll use Σ-notation in Riemann sums
(briefly)

then, we'll use it <u>heavily</u> for series.

Write Riemann Sums in Σ notation

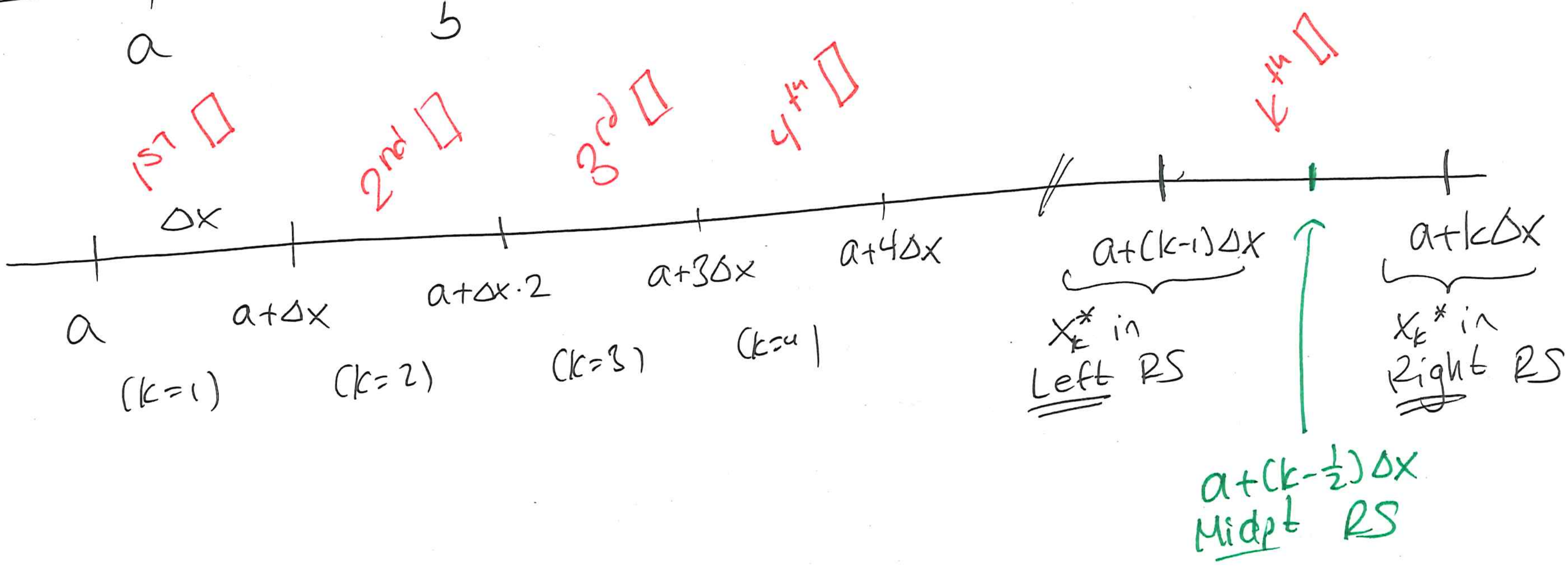


n pieces

Width: $\Delta x = \dfrac{b-a}{n}$

Height: $f(x^*)$

which x^*?



Riemann Sum:

$$(\text{Area of } 1^{st} \square) + (\text{Area of } 2^{nd} \square) + \cdots + (\ n^{th} \square)$$

$$= \underbrace{\Delta x}_{\text{base}} \cdot \underbrace{f(x_1^*)}_{\text{height}} + \Delta x \cdot f(x_2^*) + \cdots + \Delta x\, f(x_n^*)$$

$$= \sum_{k=1}^{n} \Delta x\, f(x_k^*)$$

Left RS: x_k^*: $a+(k-1)\Delta x$

Right RS: x_k^*: $a+k\Delta x$

Midpt RS: x_k^*: $a+(k-\frac{1}{2})\Delta x$

General Form of Riemann Sums

Can efficiently write RS w <u>lots</u> of $\square$s!

ex Right RS of $y = x^2 + x$, $1 \le x \le 6$, using 100 subintervals.

$a = 1 \quad b = 6 \quad n = 100$

$\Delta x = \frac{6-1}{100} = \frac{5}{100} = \frac{1}{20}$

$f(x) = x^2 + x$

$$\sum_{k=1}^{n} \Delta x \, f(a + k\Delta x)$$

$$= \sum_{k=1}^{100} \frac{1}{20} \cdot f(\underbrace{1 + k \cdot \tfrac{1}{20}}_{x_k^*})$$

$$= \sum_{k=1}^{100} \frac{1}{20} \cdot \Big[\underbrace{\left(1 + k \cdot \tfrac{1}{20}\right)^2 + \left(1 + k \cdot \tfrac{1}{20}\right)}_{x^2 + x} \Big]$$

Detour: evaluating sums

① $\displaystyle\sum_{k=1}^{n} C = C + C + C + \cdots + C = n \cdot C$

C ← constant

② $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$

$\displaystyle = \frac{n(n+1)}{2}$

p340 more (proofs get complicated)

③ $\displaystyle\sum_{k=1}^{n} k^2 = 1 + 4 + 9 + 16 + 25 + \cdots = \frac{n(n+1)(2n+1)}{6}$

④ $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

$1+2+3+4+\cdots+97+98+99+100$

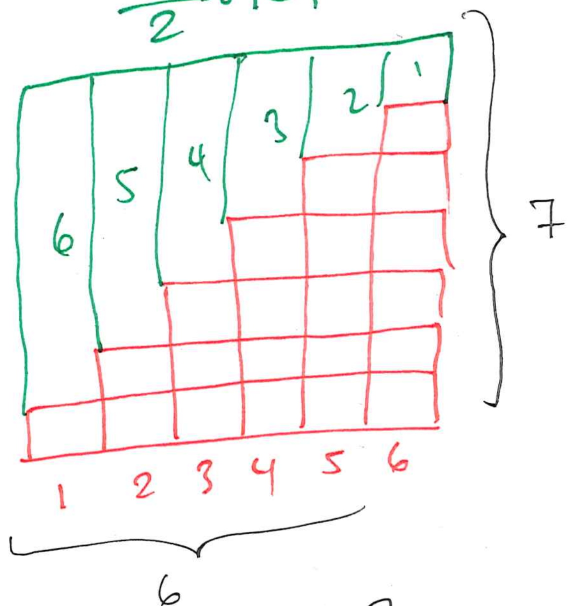
101

101

101

101

$\frac{100}{2} \cdot 101$



squares: $6 \cdot 7$

red squares: $\frac{6 \cdot 7}{2}$

Returning:

Manipulate into a form I can evaluate eg $\sum k$, $\sum k^2$

$$\sum_{k=1}^{100} \frac{1}{20}\left[\underbrace{(1+\tfrac{1}{20}k)^2} + (1+\tfrac{1}{20}k)\right]$$

$$= \frac{1}{20}\left[\sum_{k=1}^{100}\left(\overbrace{1+\tfrac{2}{20}k+\tfrac{1}{400}k^2} + 1+\tfrac{1}{20}k\right)\right]$$

$$= \frac{1}{20}\left[\sum_{k=1}^{100}\left(2+\tfrac{3}{20}k+\tfrac{1}{400}k^2\right)\right]$$

$$= \frac{1}{20}\left(\sum_{k=1}^{100} 2 + \sum_{k=1}^{100}\tfrac{3}{20}k + \sum_{k=1}^{100}\tfrac{1}{400}k^2\right)$$

$$= \frac{1}{20}\left(200 + \frac{3}{20}\boxed{\sum_{k=1}^{100} k} + \frac{1}{400}\boxed{\sum_{k=1}^{100} k^2}\right)$$

Use formulas from p 340

$$= \frac{1}{20}\left(200 + \frac{3}{20}\cdot\frac{100(101)}{2} + \frac{1}{400}\cdot\frac{100(101)(201)}{6}\right)$$

We just added 100 different areas!

ex Left RS:

$$\sum_{k=1}^{n} \Delta x \cdot f(a+(k-1)\Delta x)$$

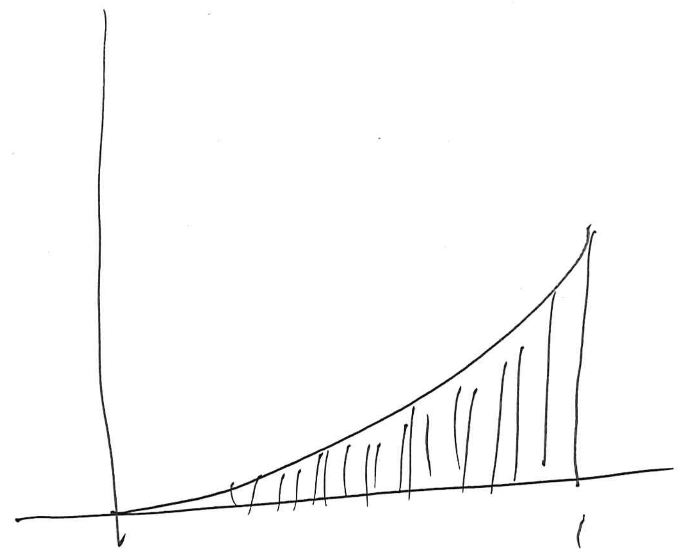
$\Delta x = \frac{b-a}{n}$

$f(x) = x^3$

$a=0, \quad b=1$

$n=50$

$\Delta x = \frac{1-0}{50} = \frac{1}{50}$



$$= \sum_{k=1}^{50} \frac{1}{50}\left(0+(k-1)\cdot\frac{1}{50}\right)^3$$

$$= \sum_{k=1}^{50} \frac{1}{50}\left(\frac{k-1}{50}\right)^3$$

$$= \frac{1}{50}\sum_{k=1}^{50}\left(\frac{1}{50}\right)^3 (k-1)^3$$

$$= \frac{1}{50^4}\sum_{k=1}^{50}(k-1)^3$$

ONE WAY:

$$= \frac{1}{50^4}\sum_{k=1}^{50}(k^3-3k^2+3k-1)$$

$$= \frac{1}{50^4}\left(\sum_{k=1}^{50} k^3 - 3\underbrace{\sum_{k=1}^{50} k^2}_{} + 3\sum_{k=1}^{50} k - \sum_{k=1}^{50} 1 \right)$$

Formulas

$$= \frac{1}{50^4}\left(-3\,\frac{50\cdot 51\cdot 101}{6} + \frac{50^2\cdot 51^2}{4} + 3\cdot\frac{50\cdot 51}{2} - 50\right)$$

ANOTHER WAY:

$$\frac{1}{50^4}\underbrace{\sum_{k=1}^{50}(k-1)^3}_{\underbrace{0^3}_{\text{ignore}}+1^3+2^3+\cdots+49^3} = \frac{1}{50^4}\sum_{k=1}^{49}k^3 = \frac{1}{50^4}\cdot\frac{n^2(n+1)^2}{4}$$

formula

$n=49$

$$= \frac{1}{50^4}\cdot\frac{49^2\cdot 50^2}{4}$$

Next: $\lim\limits_{n\to\infty} \square^n$,