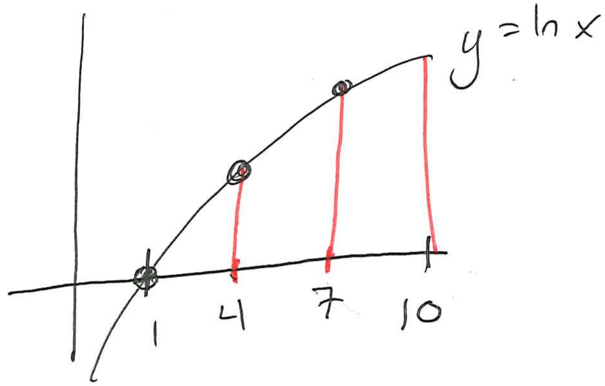
Riemann Sums (approximate area under curve)

3 kinds: Left, Right, Midpoint

(ex) $f(x) = \ln x$, $[1, 10]$, $n = 3$:



Base of rectangles: $\frac{10-1}{3} = \frac{9}{3} = 3 = \Delta x$

Left RS: heigh of rectangles

0,	$\ln 4$,	$\ln 7$
1st ▯	2nd ▯	3rd ▯

Left RS: $3 \times 0 + 3\ln 4 + 3\ln 7$

$= \boxed{3\ln(28)}$

Right RS:

heights

1st ▯ $\ln(4)$

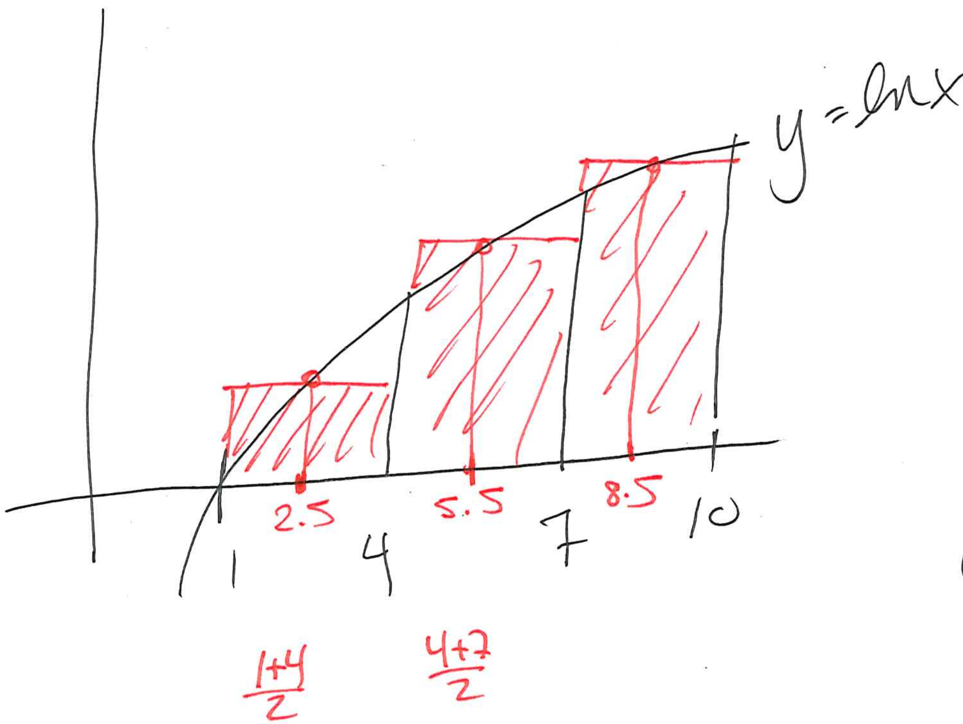
2nd ▯ $\ln(7)$

3rd ▯ $\ln 10$

So Right RS: $3\ln 4 + 3\ln 7 + 3\ln 10$

$= 3\ln(280)$

Midpt :



RS:

$3\ln(2.5) + 3\ln(5.5) + 3(8.5)$

We need better notation for sums

Detour: Σ notation

will use now

will use <u>heavily</u> in Series

a, b: bounds
k: index

$$\sum_{k=a}^{b} f(k)$$

summands

Sigma
"sum"

ex:

$$\sum_{k=3}^{6} (2k+5) = \underset{(k=3)}{(6+5)} + \underset{(k=4)}{(8+5)} + \underset{(k=5)}{(10+5)} + \underset{(k=6)}{(12+5)}$$

$$= 20 + 36 = 56$$

ex:

$$\sum_{k=5}^{8} k \cdot 10 \qquad \sum_{k=3}^{7} 8 \qquad \sum_{k=1}^{4} (k^2 - k)$$

ex) Write in Σ-notation

Think:
$\sum_{k=1}^{3} 2k = 2+4+6$ (evens)

$\sum_{k=1}^{3} (2k+1) = 3+5+7$ (odds)

(1) $5+7+9+11+13$

$\sum_{k=2}^{6} (2k+1)$; Another way: $\sum_{k=0}^{4} (2k+5)$

(2) $3.5 + 6.5 + 9.5 + 12.5 + 15.5$

$\sum_{k=1}^{5} \left(3k+\frac{1}{2}\right) = \underset{(k=1)}{3.5} + \underset{(k=2)}{6.5} + \underset{(k=3)}{9.5}$ etc

(3) $\frac{-1}{2} + 1 - 2 + 4 - 8 + 16 - 32$

$-(2^{-1})$ 2^0 $2=2^1$ $4=2^2$ $8=2^3$ $16=2^4$ $32=2^5$

$\boxed{\sum_{k=-1}^{5} (-2)^k} = \sum_{k=-1}^{5} (-1)^k 2^k$

Think:
$\sum_{k=-1}^{5} 2^k = 2^{-1}+2^0+2^1+2^2+2^3+2^4+2^5$

$= \frac{1}{2}+1+2+4+8+16+32$

Close!

$(-1)^1 = -1$
$(-1)^2 = +1$
$(-1)^3 = -1$ etc

$$\sum_{k=5}^{8} k \cdot 10 = \underset{(k=5)}{50} + \underset{(k=6)}{60} + \underset{(k=7)}{70} + \underset{(k=8)}{80}$$

$$= 130 \times 2 = 260$$

$$\sum_{k=3}^{7} 8 = \underset{(k=3)}{8} + \underset{(k=4)}{8} + \underset{(k=5)}{8} + \underset{(k=6)}{8} + \underset{(k=7)}{8}$$

$$= 40$$

$$\sum_{k=1}^{4} (k^2 - k) = (1^2 - 1) + (2^2 - 2) + (3^2 - 3) + (4^2 - 4)$$

$$= 0 + 2 + 6 + 12 = \boxed{20}$$

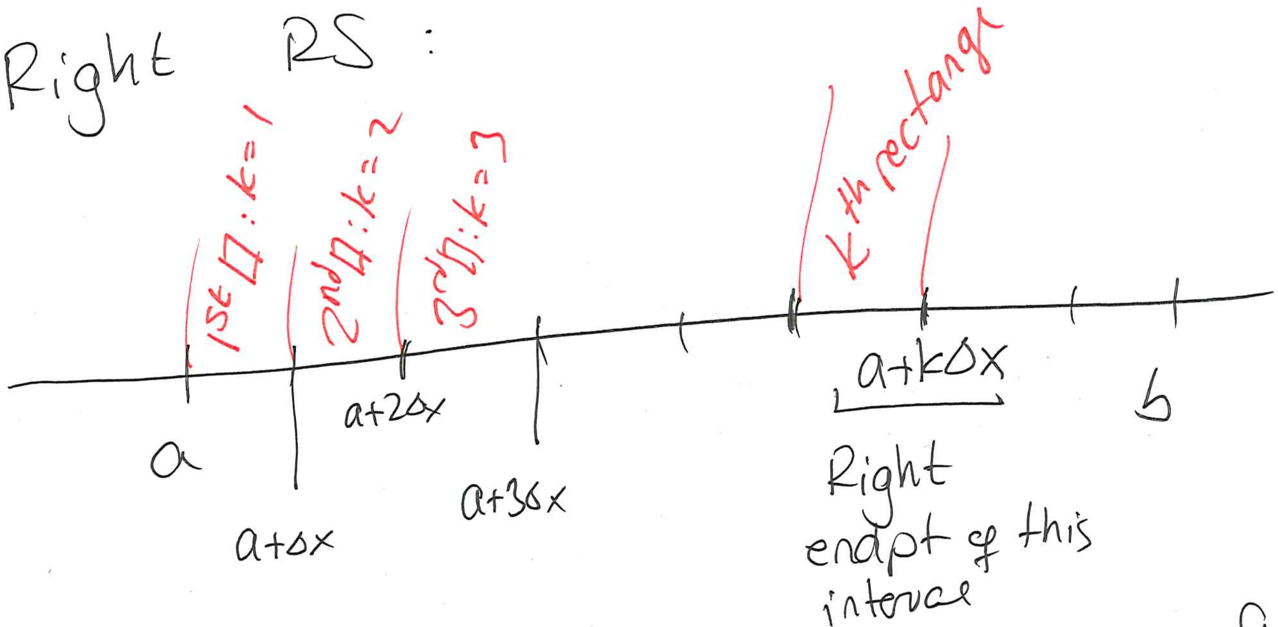
Writing Riemann Sums in Σ notation (compact notation)

Right RS :

n subintervals

width of □

$\Delta x = \frac{b-a}{n}$

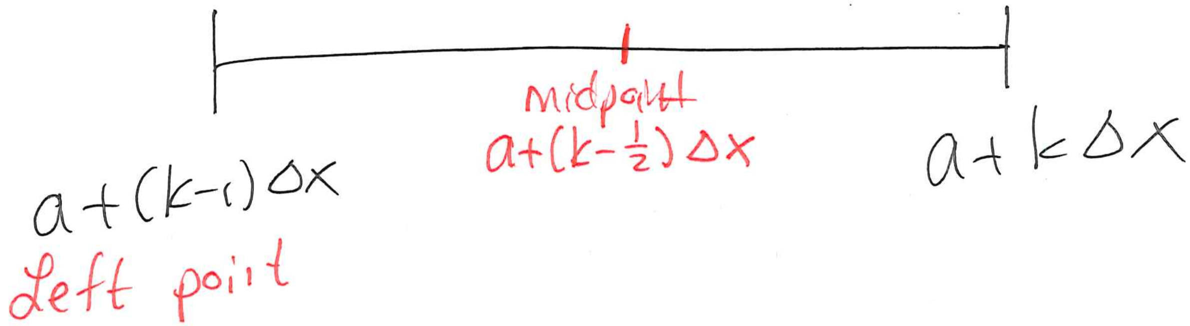


$\rightarrow x_k^*$ (how we find height of □)

k^{th} □ area: $\underbrace{\Delta x}_{b}\ \underbrace{f(a+k\Delta x)}_{h}$

Right RS: $\boxed{\sum_{k=1}^{n} \Delta x \cdot f(a+k\Delta x)}$ where $\Delta x = \frac{b-a}{n}$

k^{th} rectangle



Left RS: $\displaystyle\sum_{k=1}^{n} \Delta x \cdot f(a+(k-1)\Delta x)$

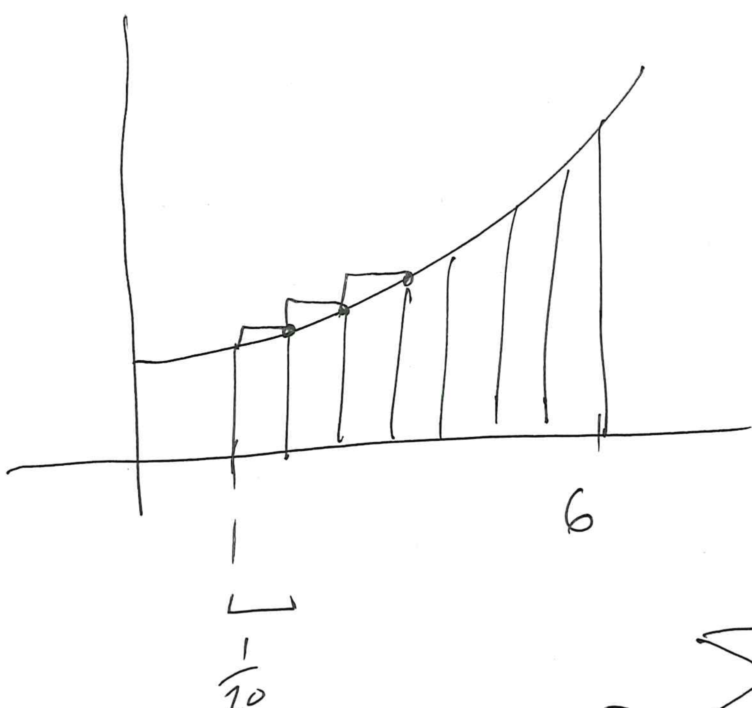
Midpt RS: $\displaystyle\sum_{k=1}^{n} \Delta x\, f(a+(k-\tfrac{1}{2})\Delta x)$

ex) Right Riemann Sum approximating area under x^2+x from $x=1$ to $x=6$ using 100 subintervals.

$n=100$

$a=1$

$\Delta x = \frac{b-a}{n} = \frac{6-1}{100} = \frac{1}{20}$



$$\sum_{k=1}^{n} \Delta x\, f(a+k\Delta x)$$

$$\sum_{k=1}^{100} \frac{1}{20} \cdot \underbrace{f\left(1+k\left(\tfrac{1}{20}\right)\right)}_{\text{height}}$$

$$= \sum_{k=1}^{100} \frac{1}{20} \Big[\underbrace{\left(1+k\cdot\tfrac{1}{20}\right)^2 + \left(1+k\cdot\tfrac{1}{20}\right)}_{x^2+x} \Big]$$

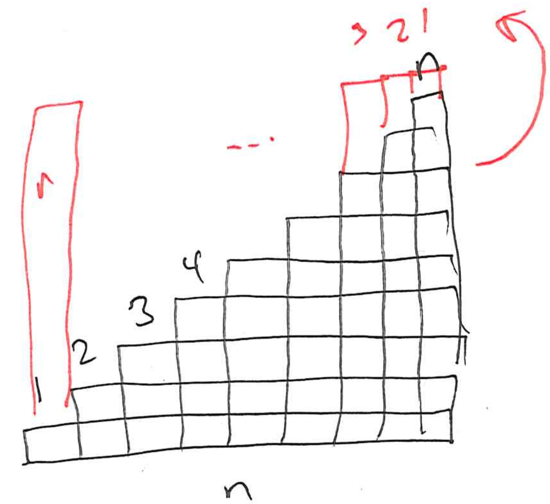
Detour: Evaluating sums (p 340)

FACTS: $\displaystyle\sum_{k=1}^{n} C = n \cdot C$ ← constant

$$\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1+4+9+16+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$



Area: $n(n+1)$

Area blacks: $\dfrac{n(n+1)}{2}$

Back to Riemann Sum:

$$\sum_{k=1}^{100} \frac{1}{20}\left[\left(1+\tfrac{1}{20}k\right)^2 + \left(1+\tfrac{1}{20}k\right)\right]$$

Goal: "simplify" until I can use facts from p 340

Need: algebra using Σ-notation ! :) ♡

FACT: $5+17 = 17+5$ Addition is commutative

$$\sum (A+B) = (A+B)+(A+B)+\cdots+(A+B)$$
$$= (A+A+A+\cdots+A) + (B+B+B+\cdots+B)$$
$$= \left(\sum A\right) + \left(\sum B\right)$$

FACT: Distribution $\quad \frac{1}{20}(1+2+3) = \frac{1}{20}(1) + \frac{1}{20}(2) + \frac{1}{20}(3)$

$$\sum_{k=1}^{4} c\cdot k = c(1)+c(2)+c(3)+c(4) =$$
$$c[1+2+3+4] = c\sum_{k=1}^{4} k$$

Riemann Sum:

$$\sum_{k=1}^{100} \frac{1}{20}\left[\left(1+\frac{1}{20}k\right)^2 + 1+\frac{1}{20}k\right]$$

$$= \frac{1}{20}\sum_{k=1}^{100}\left(1+\frac{2}{20}k+\frac{1}{400}k^2+1+\frac{1}{20}k\right)$$

$$= \frac{1}{20}\sum_{k=1}^{100}\left(2+\frac{3}{20}k+\frac{1}{400}k^2\right)$$

commutativity

$$= \frac{1}{20}\left(\sum_{k=1}^{100}2+\sum_{k=1}^{100}\frac{3}{20}k+\sum_{k=1}^{100}\frac{1}{400}k^2\right)$$

factor out

$$= \frac{1}{20}\left(200+\frac{3}{20}\sum_{k=1}^{100}k+\frac{1}{400}\sum_{k=1}^{100}k^2\right)$$

$$= \boxed{\frac{1}{20}\left(200+\frac{3}{20}\left(\frac{100\cdot 101}{2}\right)+\frac{1}{400}\left(\frac{100\cdot 101\cdot 201}{6}\right)\right)}$$

What if we wanted ∞ many slices?

Right RS

n slices: $\displaystyle\sum_{k=1}^{n} \Delta x\, f(a+k\Delta x)$

$$\Delta x = \frac{b-a}{n}$$

∞ many: $\displaystyle\lim_{n\to\infty}\left(\sum_{k=1}^{n} \Delta x\, f(a+k\Delta x)\right)$

ex $f(x)=x^2$ $[a,b]=[0,10]$

$$\Delta x = \frac{10-0}{n} = \frac{10}{n}$$

$$a+k\Delta x = 0+k\left(\frac{10}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{10}{n} \underbrace{\left(k\,\frac{10}{n}\right)^2}_{f(x)=x^2}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{10}{n} (k^2*100/n^2)$$

$$= \lim_{n\to\infty}\left[\frac{10}{n}\left(\sum_{k=1}^{n} k^2 + \frac{20}{n}\sum_{k=1}^{n} k + \sum_{k=1}^{n}\frac{100}{n}\right)\right]$$

oops--will fix next class!

$$= \lim_{n\to\infty}\left[\frac{10}{n}\left[\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{20}{n}\cdot\frac{n(n+1)}{2} + n\left(\frac{100}{n}\right)\right]\right]$$

eval limit as in Calc I