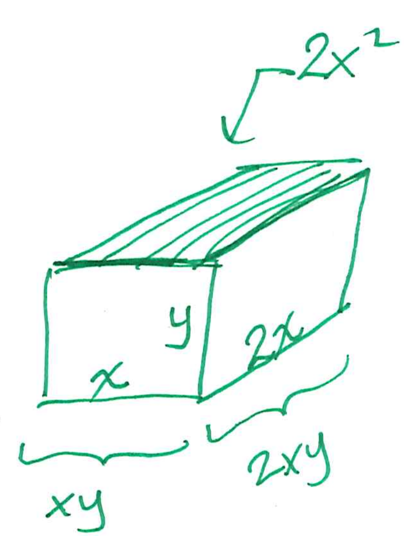
Midterm info on course website

No formula sheet

Expect a variety of difficulties
of questions
(roughly ordered from straightforward
to nonstandard)

Ch 12.9 Lagrange Multipliers



(ex) A rectangular box needs:

- Volume of 72 cubic cm
- Width twice length
- Min possible surface area

$f(x,y)$ "objective" : want to mininimize

surface area

$$f(x,y) = 2xy + 4xy + 4x^2$$

$$\boxed{f(x,y) = 6xy + 4x^2}$$

$g(x,y) = 0$ "constraint"

Only care about dimensions (x,y) that give volume 72.

Volume: $(x)(2x)y = 2x^2y = 72$

$\boxed{g(x,y) = 2x^2y - 72} = 0$

$$\textcircled{1} \quad \underbrace{6y + 8x}_{f_x} = \lambda \underbrace{(4xy)}_{g_x} \rightarrow \boxed{\lambda = \frac{6y+8x}{4xy}} \quad \text{or} \quad \cancel{4xy = 0}$$

$$\textcircled{2} \quad \underbrace{6x}_{f_y} = \lambda \underbrace{(2x^2)}_{g_y}$$

$$\textcircled{3} \quad 2x^2y - 72 = 0$$

If $4xy = 0$, then $x = 0$ or $y = 0$

③ $0 - 72 = 0$ FALSE

NO POINTS TO CONSIDER

$$\lambda = \frac{6y+8x}{4xy} = \frac{3y+4x}{2xy}$$ ① true

② $$6x = \underbrace{\left(\frac{3y+4x}{\cancel{2}xy}\right)}_{\lambda} \cdot \cancel{2}x^2$$

$$6\cancel{x} = \frac{3y+4x}{y} \cdot \cancel{x}$$

$$6y = 3y+4x$$

$$3y = 4x$$

$$\boxed{y = \frac{4}{3}x}$$

③ $$2x^2\underbrace{\left(\frac{4}{3}x\right)}_{y} = 72$$

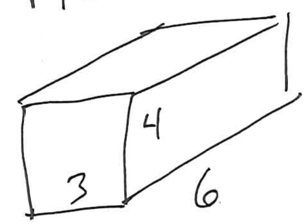
$$\frac{\cancel{8}}{3}x^3 = \cancel{8}\cdot 9$$

$$x^3 = 27$$

$$\boxed{x=3}$$

$$\boxed{y=4}$$

Absolute min:



ex What is the largest value of x on the ellipse

$$x^2 - 2xy + 5y^2 = 1 \quad ?$$

Constraint: $\underbrace{x^2 - 2xy + 5y^2 - 1}_{g(x,y)} = 0$

Objective: $f(x,y) = x$

$$\left.\begin{array}{l} ① \ f_x = \lambda g_x \\ ② \ f_y = \lambda g_y \\ ③ \ x^2 - 2y\cancel{x} + 5y^2 - 1 = 0 \end{array}\right\} \quad \left.\begin{array}{l} 1 = \lambda \cdot (2x - 2y) \\ 0 = \lambda(-2x + 10y) \\ x^2 - 2y\cancel{x} + 5y^2 = 1 \end{array}\right\} \longrightarrow \boxed{\lambda = \frac{1}{2x - 2y}} \quad \text{or} \quad \cancel{\begin{array}{c} 2x - 2y = 0 \\ (\text{FALSE}) \\ 1 = \lambda \cdot 0 \end{array}}$$

$$\lambda = \frac{1}{2x-2y}$$

(2) $0 = \lambda(-2x+10y)$

$$0 = \left(\frac{1}{2x-2y}\right)(-2x+10y)$$

$$0 = -2x+10y$$

$$\boxed{x = 5y}$$

(3) $x^2 - 2xy + 5y^2 = 1$

$$(5y)^2 - 2(5y)y + 5y^2 = 1$$

$$25y^2 - 10y^2 + 5y^2 = 1$$

$$20y^2 = 1$$

$$y^2 = \frac{1}{20}$$

$$\boxed{y = \frac{\pm 1}{\sqrt{20}}}$$

$x = \frac{5}{\sqrt{20}}$ or $x = \frac{-5}{\sqrt{20}}$

(Points to check)

$f(x,y) = x$

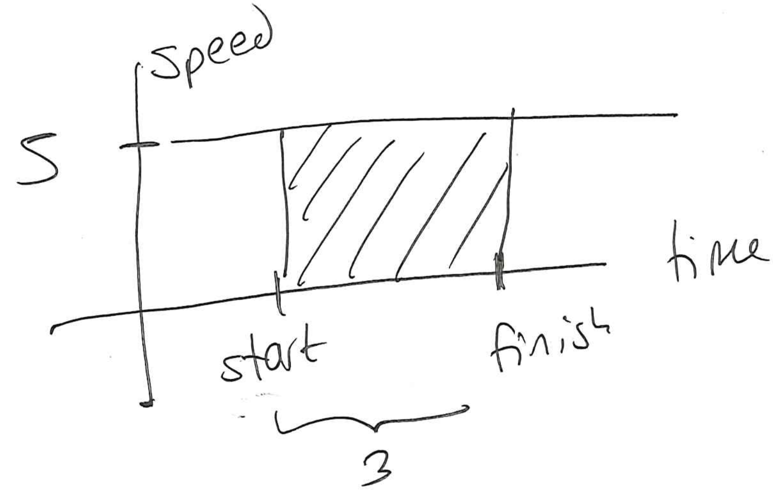
Max x on ellipse: $\frac{5}{\sqrt{20}}$

$$= \frac{5}{2\sqrt{5}} = \frac{\sqrt{5}}{2}$$

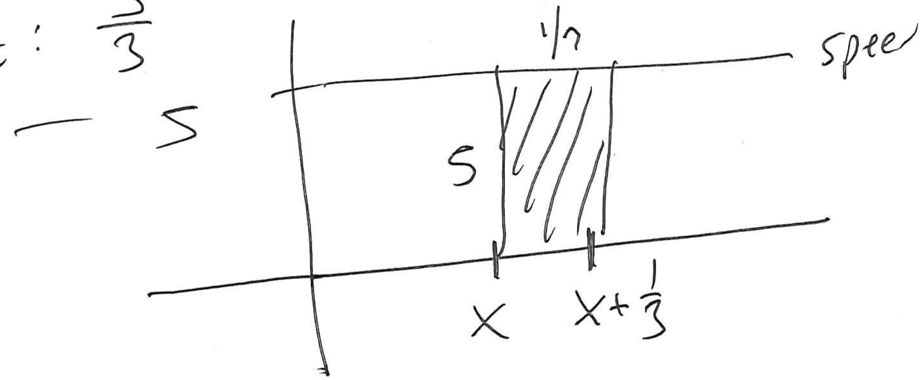
Ch 5.1 | Area under Curves

Motivation : relationship btw rate of change &
amount of change

(ex) Speed : 5 m/s
Time : 3 sec
Dist travelled : 15 m



(ex) Speed : 5 m/s
Time : $\frac{1}{3}$ sec
Dist : $\frac{5}{3}$

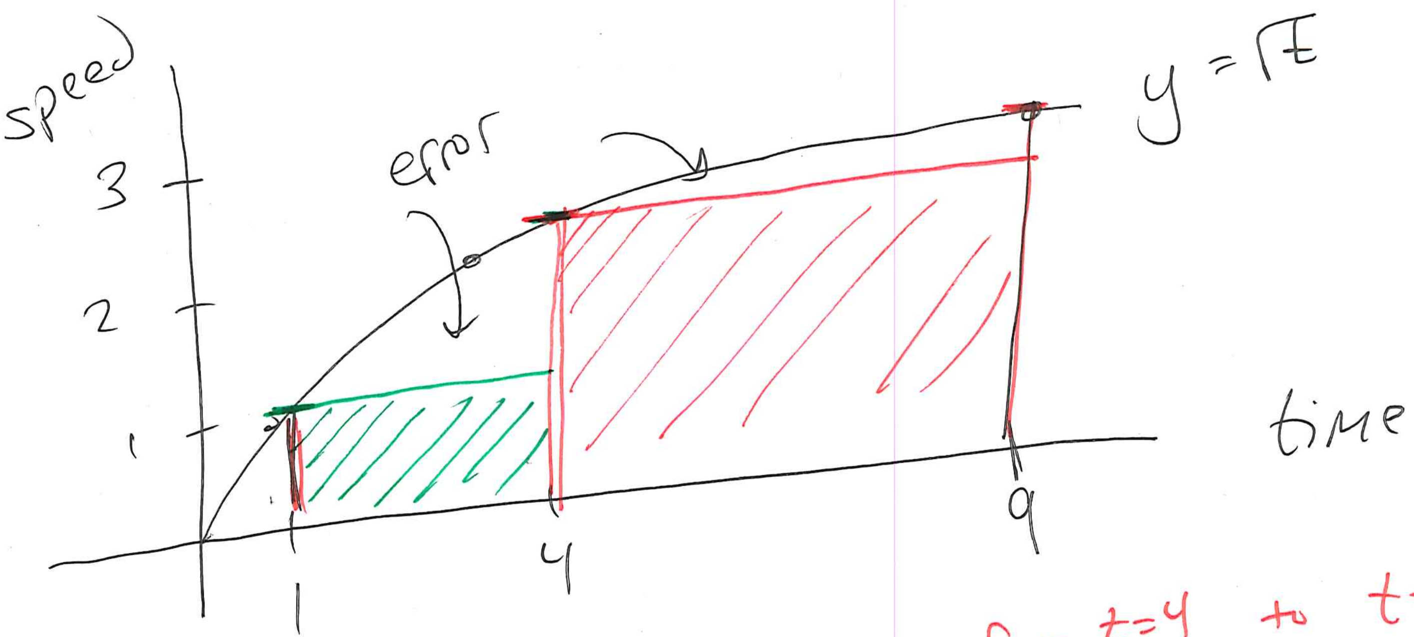


ex Speed: at time t, $\sqrt{t}$

$1 \le t \le 9$

Dist: still area under curve

Now: harder to calculate



Let's estimate:

3 seconds from $t=1$ to $t=4$

Appox speed: 1 m/s

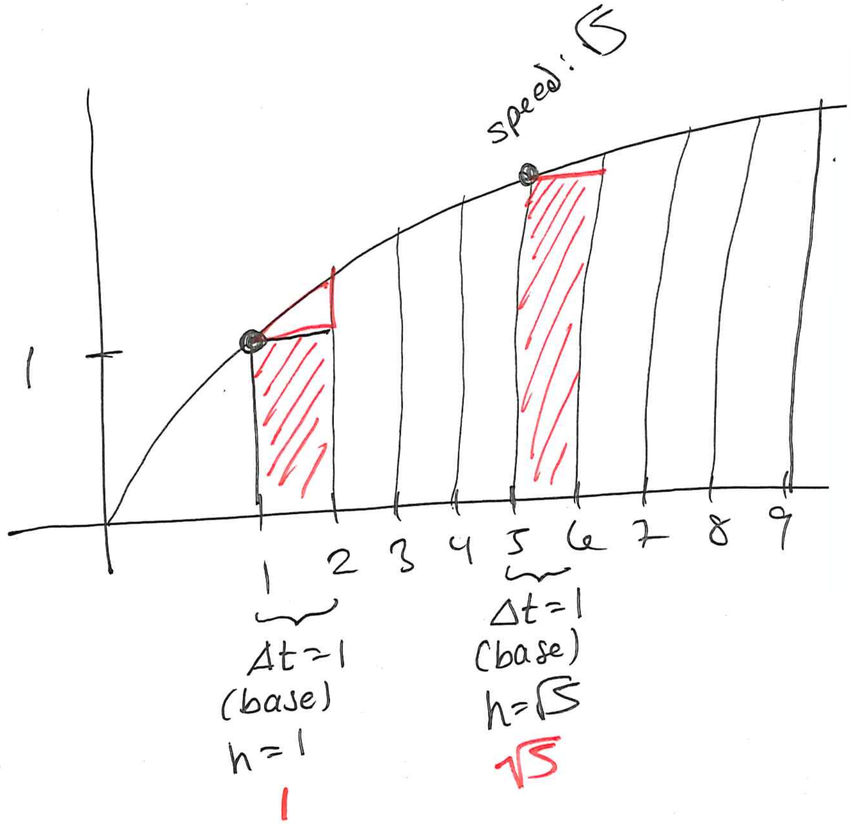
Dist $\approx 3 \cdot 1 = 3$ m

5 sec from $t=4$ to $t=9$

Approx: speed $\approx$ 2 m/s

Dist $\approx 5 \cdot 2 = 10$ m

Ttl dist $\approx$ $\boxed{13 \text{ m}}$



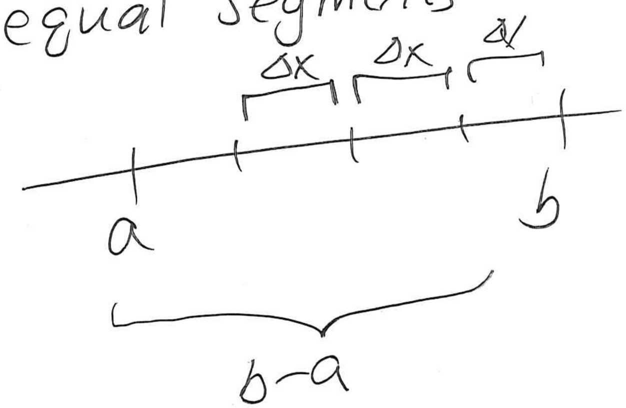
$y = \sqrt{x}$ Better Approx

Riemann Sums

Setup: Area under curve $y = f(x)$ on $[a, b]$

Regular partition: cut $[a, b]$ into equal segments

(not "general") length of a segment: Δx
(base of rectangle)



Using n rectangles:

$$\Delta x = \frac{b-a}{n}$$

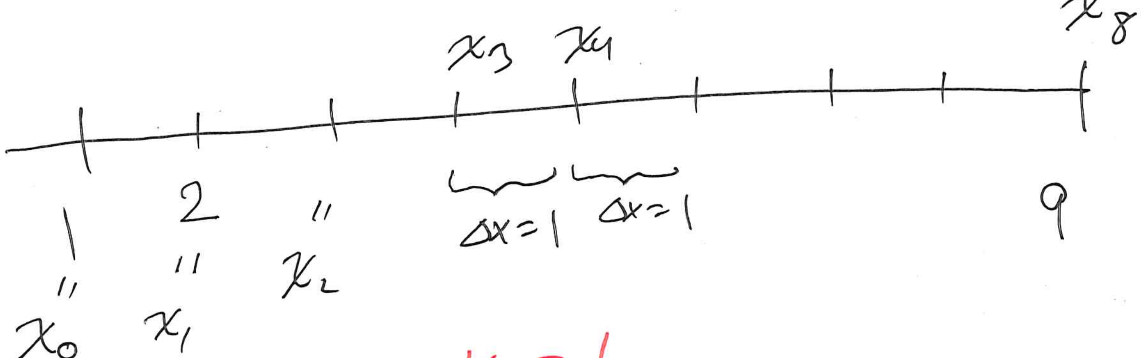
"Height" of rectangle at point x:
$f(x)$

The point we plug in for the k^{th} rectangle, we write:

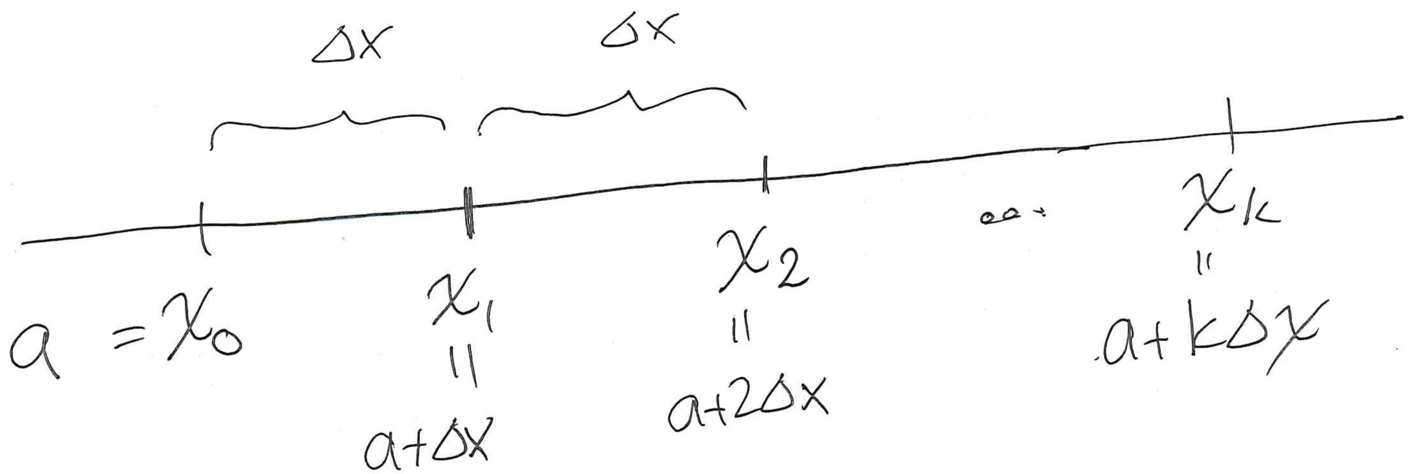
$$x_k^*$$

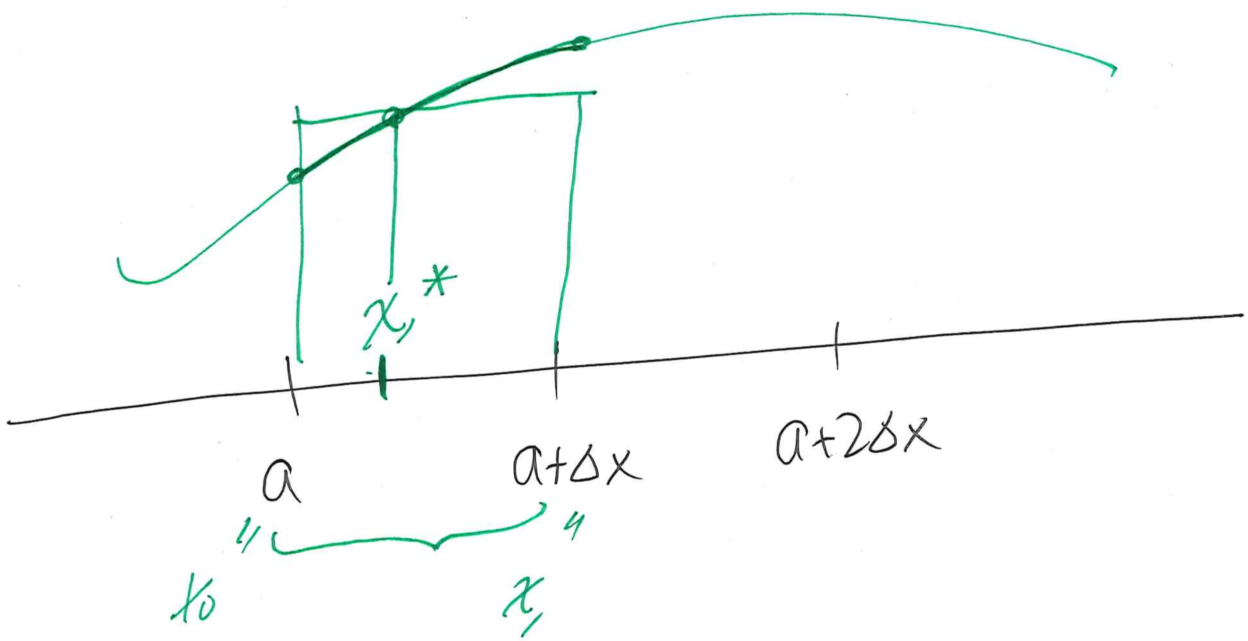
x_k : "grid points" : where rectangles start / end

$n = 8$
$\Delta x = \frac{9-1}{8} = 1$



grid points: $x_0 = 1$
$x_1 = 2$
etc

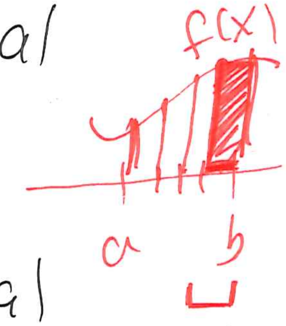




1st rectangle

Area: $\Delta x \cdot f(x_1^*)$

Suppose f is defined on a closed interval $[a, b]$, which is divided into n equal subintervals of length Δx.



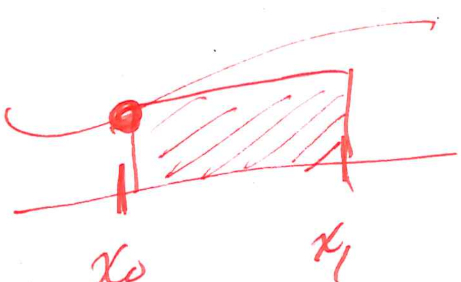
Δx - base of rectangle

If x_k^* is <u>any</u> point in the k^{th} subinterval $[x_{k-1}, x_k]$, for $k = 1, \dots, n$, then

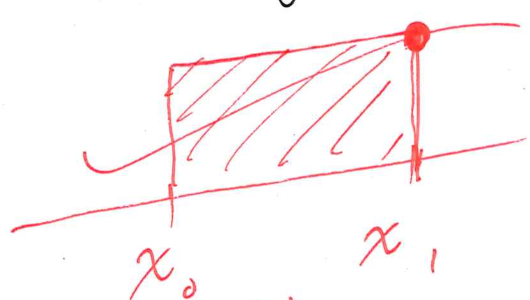
$$f(x_1^*)\Delta x + \underbrace{f(x_2^*)}_{h}\,\underbrace{\Delta x}_{\text{base}} + \cdots + \underbrace{f(x_n^*)}_{\text{height}}\,\underbrace{\Delta x}_{\text{base}}$$

is called a <u>Riemann Sum</u>

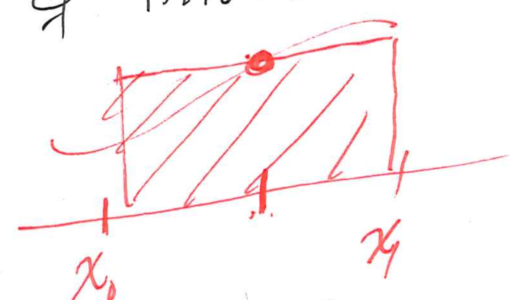
It's a left/right/midpoint Riemann sum if x_k^* is the left/right/midpt of interval.



Left

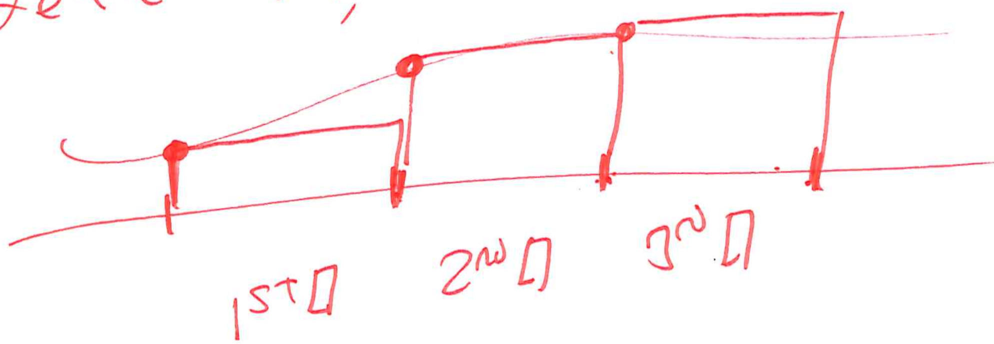


Right

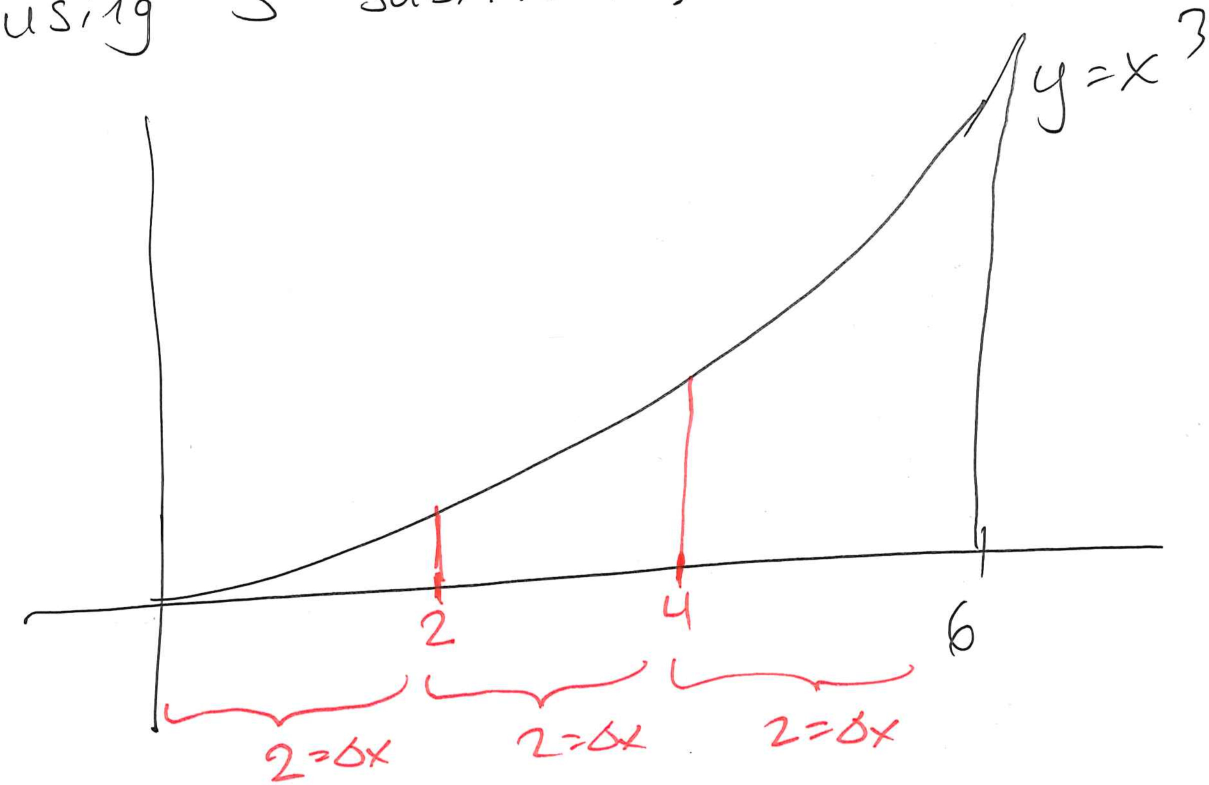


Midpoint

Left RS, $n=3$



(ex) Set up Riemann Sum to approx area under $f(x)=x^3$ from $x=0$ to $x=6$, using 3 subintervals.



$n=3$

$\Delta x = \frac{6-0}{3} = 2$

Grid points:

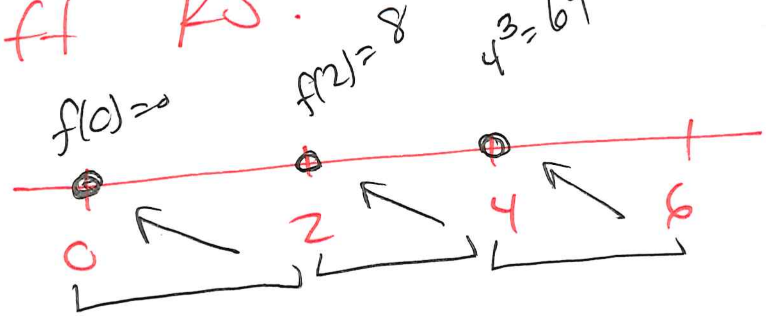
$x_0 = 0$

$x_1 = 2$

$x_2 = 4$

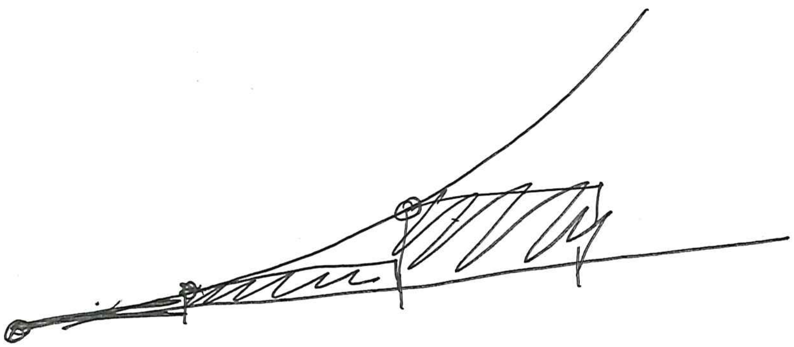
$x_3 = 6$

Left RS:



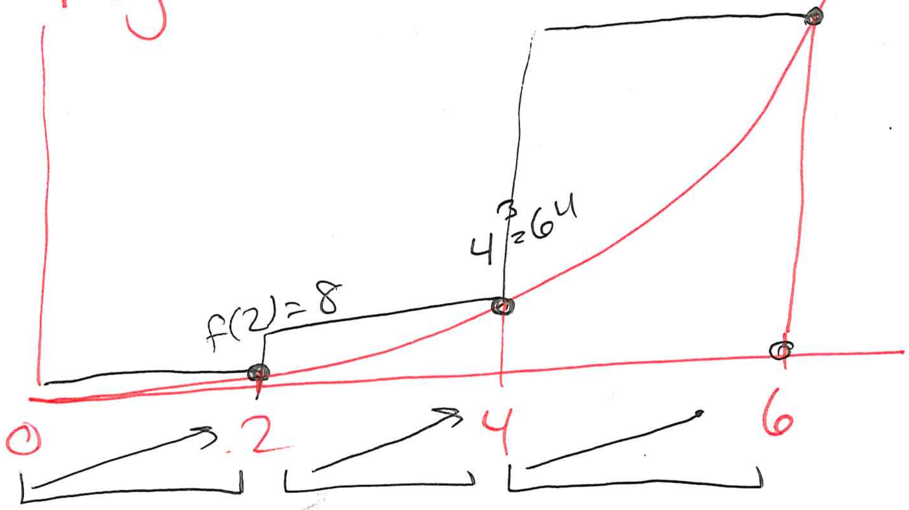
RS:

$$\underbrace{2}_{\Delta x}\cdot \overset{f(x_1^*)}{0} + \underbrace{2}_{\Delta x}\cdot \overset{f(x_2^*)}{8} + \underbrace{2}_{\Delta x}\cdot \overset{f(x_3^*)}{64}$$



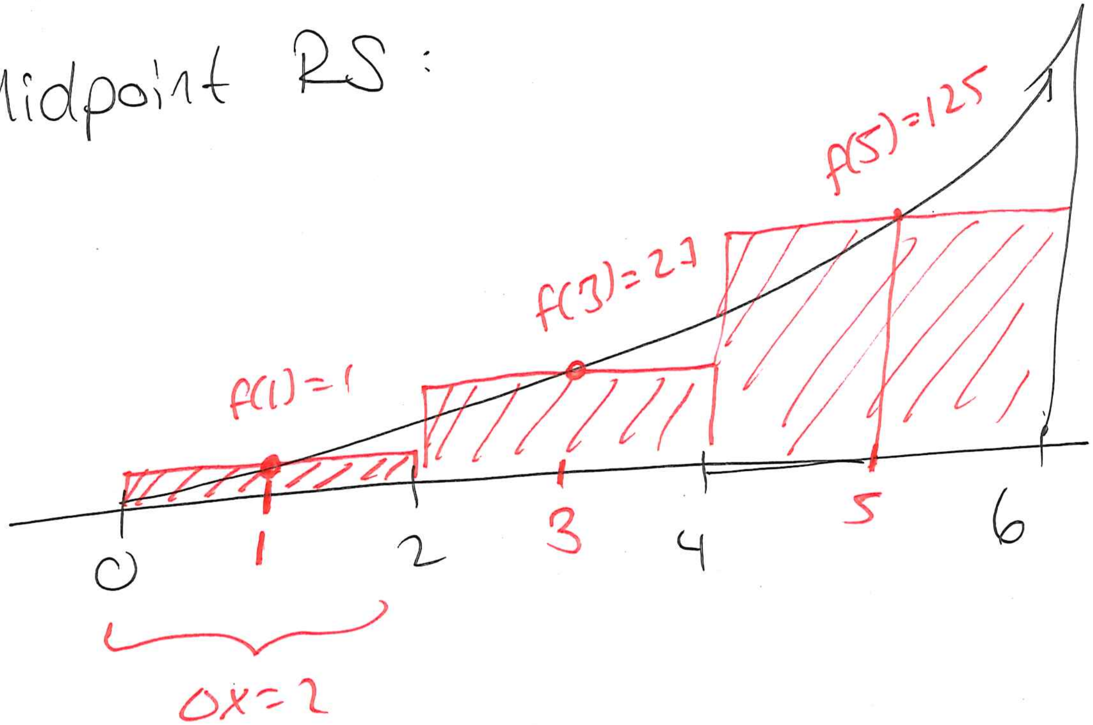
Right RS

$y=x^3$



$$\overset{\Delta x}{2}\cdot \overset{f(x_1^*)}{8} + 2\cdot 64 + 2\cdot 216$$

Midpoint RS:



$$\boxed{2\cdot 1+2\cdot 27+2\cdot 125}$$

(ex)

A car driver notices following speeds:

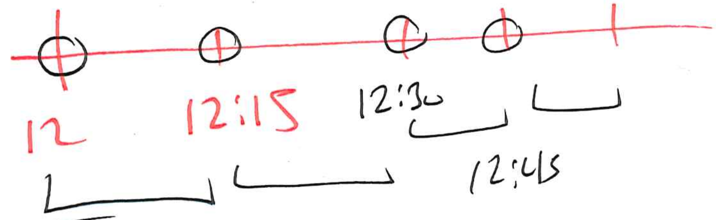
time	12:00	12:15	12:30	12:45	1:00
speed	60	80	100	100	40

Use a Riemann Sum to approx distance driver moved from 12:00 to 1:00

- how many intervals? 4 or 2
- left / right / midpoint ?

Left RS, $n=4$:

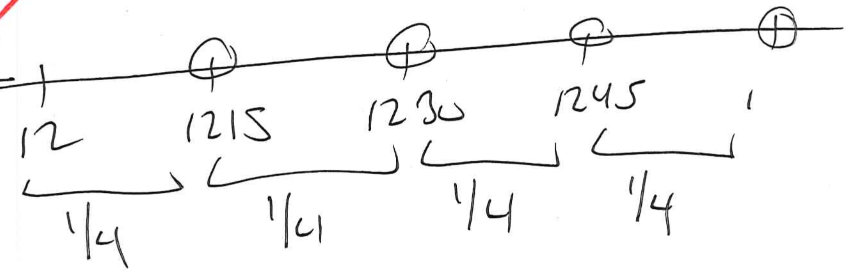
$\Delta x = \frac{1}{4}$ (hr)



$$\boxed{\frac{1}{4}\cdot 60 + \frac{1}{4}\cdot 80 + \frac{1}{4}\cdot 100 + \frac{1}{4}\cdot 100}$$

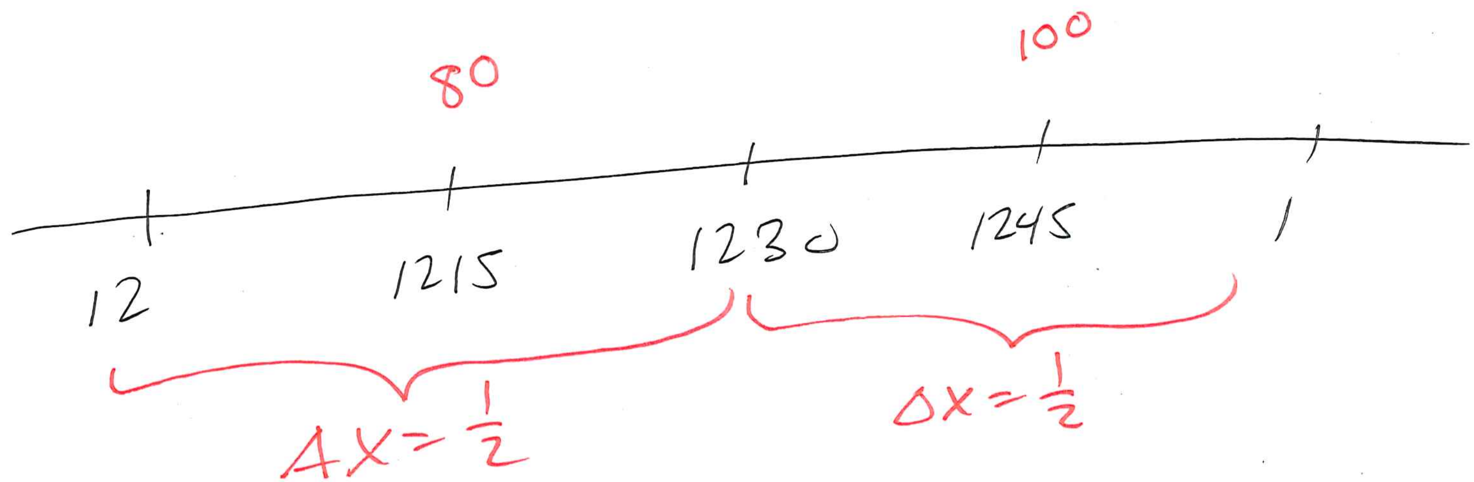
$15 + 20 + 50 = 85$ km

Right RS:



$$\boxed{\frac{1}{4}\cdot 80 + \frac{1}{4}\cdot 100 + \frac{1}{4}\cdot 100 + \frac{1}{4}\cdot 40}$$

Midpoint Riemann Sum:



$\boxed{\frac{1}{2}\cdot 80 + \frac{1}{2}\cdot 100} = 90$ km

60 kph, 80 kph

If you averaged say

Fine approx — not a Riemann Sum