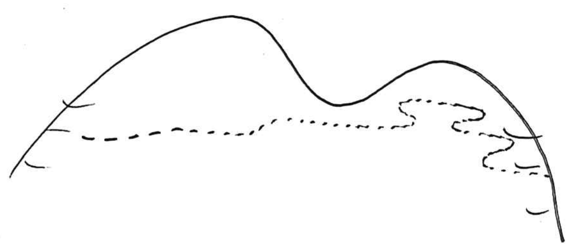
Problem #1

More investment leads to more returns, but you only have \$1000 to invest.

If $f(x,y)$ is your expected profit from investing \$$x$ in Space-X and \$$y$ in Yelp, optimize $f(x,y)$ subject to the constraint $x+y=1000$.

Problem #2



The height of a mountain at GPS coordinates (x,y) is given by the function $f(x,y)$.

A trail only traces a path along part of the mountain, say those positions (x,y) with $g(x,y)=0$.

What is the highest point reached by the trail?

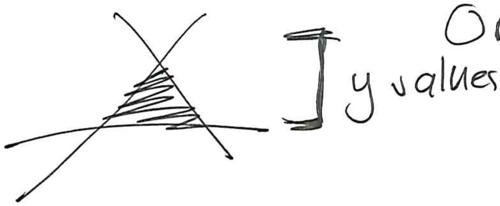
Connection

Although the context is different, the questions are fundamentally asking the same thing.

Till now, we used "plugging in"

example: Optimize $f(x,y) = x^2y + 5$

over portion of boundary with $x = 1$:



On this portion: $f(x,y) = 1^2y + 5 = y + 5$

easy to find extrema of new function given bounds on y

It might be hard to "plug in" boundary:

ex: Optimize $f(x,y) = x + 2y$

over boundary: $x^3y + x^2y^2 + y^3 + xy^3 = 2$

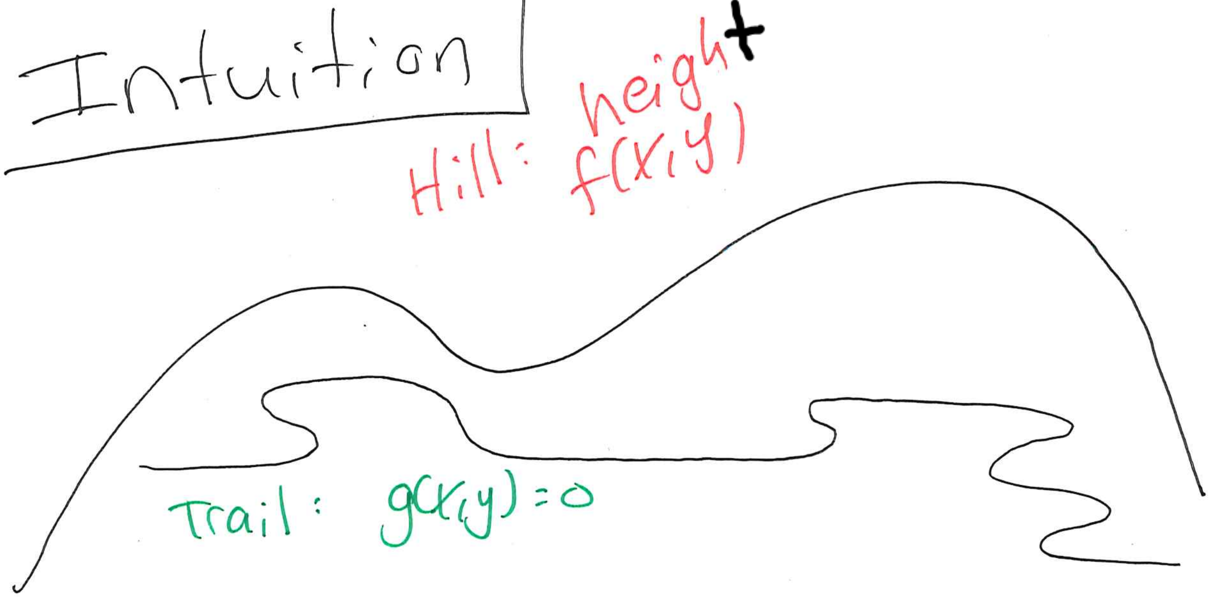
Hard to solve for x, or y!

We have another tool: Lagrange Multipliers

Lagrange Multipliers

Ch 12.9

Intuition



At highest & lowest points along trail, trail goes in a direction that is not uphill or downhill. That means:

in this direction, $z = f(x,y) = \text{constant}$.

If the trail is at local max/min: trail going same direction as a level curve.

Need a way of describing directions of level curve and trail (constraint, $g(x,y)=0$)

The gradient of $f(x,y)$ is a vector, computed:

$$\nabla f(x,y) = \langle f_x(x,y),\ f_y(x,y) \rangle$$

The gradient points in direction of steepest increase of surface, and is orthogonal to level curve.

ex Surface $f(x,y) = 2xy + x^2 + y$

$$\nabla f(x,y) = \langle \underbrace{2y+2x}_{f_x},\ \underbrace{2x+1}_{f_y} \rangle$$

At point (say) $(2,1)$:

$$\nabla f(2,1) = \langle 6, 5 \rangle$$

So, if you are standing at $(2,1)$, the steepest uphill direction is $\langle 6, 5 \rangle$.

(some fine print omitted here)

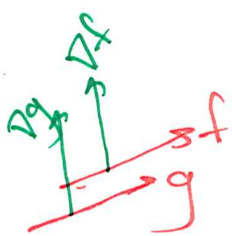
Intuition: level curve of $f(x,y)$ should be in $\underbrace{\text{same direction}}_{\text{parallel vectors}}$ as trail, $g(x,y)=0$.

parallel vectors are if they are scalar multiples of each other

$\Rightarrow$ Idea: max/min along our restricted domain, $g(x,y)=0$ should only occur if

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

for some real number λ (constant)



Ballpark Theorem: Let f be a differentiable function in a region of $\mathbb{R}^2$ that contains a smooth curve C given by $g(x,y)=0$.

If f has a local extreme value on C at (a,b), then there exists a real number λ "Lagrange (if $\nabla g(x,y) \neq 0$) multiplier" such that

$$\nabla f(a,b) = \lambda \cdot \nabla g(a,b).$$

thing we we want to max/min

f: "objective function"

"constraint function" (trail)

C: curve $g(x,y)=0$ (boundary of region)

Very restricted domain

$$\langle f_x, f_y \rangle = \lambda \langle g_x, g_y \rangle = \langle \lambda g_x, \lambda g_y \rangle$$

① $f_x = \lambda g_x$

② $f_y = \lambda g_y$

Method of Lagrange Multipliers

Setup: constraint $g(x,y)=0$

Over that constraint, find extrema of $f(x,y)$

- Find all points (x,y) such that:

$$\begin{cases} ① & f_x(x,y) = \lambda g_x(x,y) \\ ② & f_y(x,y) = \lambda g_y(x,y) \\ ③ & g(x,y) = 0 \end{cases}$$

"parallel vectors"
λ - same in both
we never really use λ -
only needs to exist

← on restricted domain

- Compare $f(x,y)$ at points you found.

Reminder (algebra fact)

Suppose $Ax = B$

True or False: $x = B/A$

Possibly: $x = B/A$

Also: maybe $A = 0$

(also $B = 0$)

Fact: $0 \cdot 13 = 0$

Fiction: $13 = 0/0$

number

not

$A \cdot B = C$

$\boxed{A = \frac{C}{B} \ \boxed{\text{or}} \ B = 0 \ \text{(also } C = 0\text{)}}$

ex Rollercoaster has height

$f(x,y) = xy + 14$ @ point (x,y)

only exists where

$x^2+y^2=18 \rightarrow g(x,y) = x^2+y^2-18=0$

$$\begin{cases} \text{①} \quad \underbrace{y}_{f_x} = \lambda \cdot \underbrace{2x}_{g_x} \longrightarrow \boxed{\lambda = \frac{y}{2x}} \text{ or } \boxed{2x=0} \\ \text{②} \quad \underbrace{x}_{f_y} = \lambda \cdot \underbrace{2y}_{g_y} \\ \text{③} \quad x^2+y^2-18=0 \end{cases}$$

also: $y = \lambda \cdot 0$

$y=0$

$x=0, y=0$

Case 1: $\boxed{x=0,\ y=0}$

③ $0^2+0^2-18=0$ FALSE

No point to consider in this case

Case 2: $\boxed{\lambda = y/2x}$

① true

② $x = \underbrace{\left(\frac{y}{\cancel{2}x}\right)}_{\lambda} \cancel{2}y$

$x^2 = y^2$ $\boxed{x = \pm y}$

③ $x^2+y^2-18=0$

$x^2+x^2=18$

$2x^2=18$

$x^2=9$

$\boxed{x=\pm 3}$

Points to Consider:

$(3, 3)$ $(-3, 3)$

$(3, -3)$ $(-3, -3)$

$f(x,y) = xy + 14$

$f(3,3) = 9+14 = 23$

$f(-3,3) = -9+14 = 5$

$f(3,-3) = -9+14 = 5$

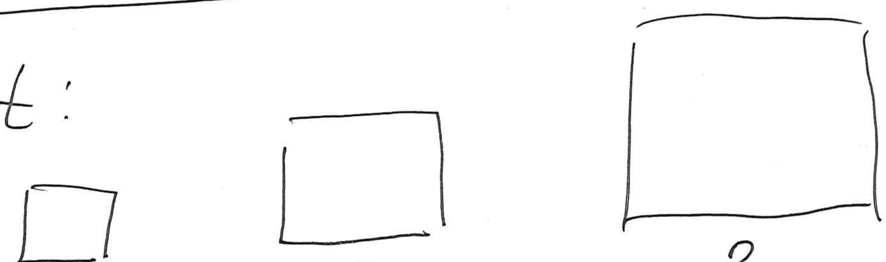
$f(-3,-3) = 9+14 = 23$

Lowest height 5 @ $(-3,3)$ $(3,-3)$

highest point height 23 @ $(3,3)$ $(-3,-3)$

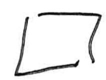
Another fact:

Squares



1 2 3

which has smallest area?



1

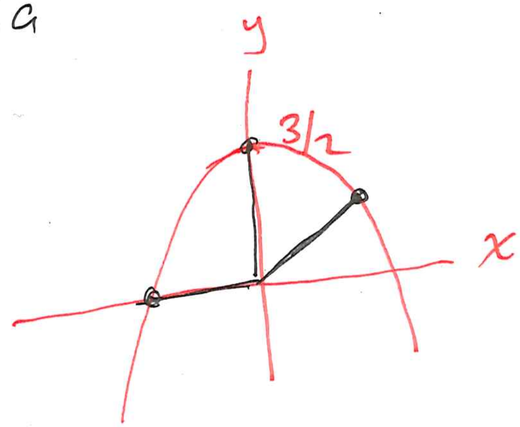
$A = x^2$

smallest value of A happens at smallest value of $x = \sqrt{A}$

TRICK: If $\sqrt{f}$ is hard to minimize, consider f.

To optimize a distance $\sqrt{x^2+y^2}$:
easier to optimize x^2+y^2
Take square root of max/min.

ex Find the point(s) on parabola
$y = 1.5 - x^2$
closest to origin.



want to minimize:
dist from point (x,y) to $(0,0)$
That is: $\sqrt{x^2+y^2}$
Sub-problem (easier): minimize x^2+y^2
Objective function: $f(x,y) = x^2+y^2$
Only care about (x,y) on parabola
Constraint: $\underbrace{y+x^2-1.5}_{g(x,y)} = 0$

$$\left.\begin{aligned} &① \quad \underbrace{2x}_{f_x} = \lambda \cdot \underbrace{2x}_{g_x} \\ &② \quad \underbrace{2y}_{f_y} = \lambda \cdot \underbrace{1}_{g_y} \\ &③ \quad y + x^2 - 1.5 = 0 \end{aligned}\right\} \to \boxed{\lambda = \frac{2x}{2x} = 1} \text{ or } \boxed{x = 0}$$

Case 1: $\lambda = 1$

① true

② $2y = 1$, so $y = 1/2$

③ $y + x^2 = 1.5$

$\frac{1}{2} + x^2 = 1.5$

$x^2 = 1$

$x = \pm 1$

Points to Consider:

$(1, 1/2)$

$(-1, 1/2)$

Case 2: $x=0$

① true

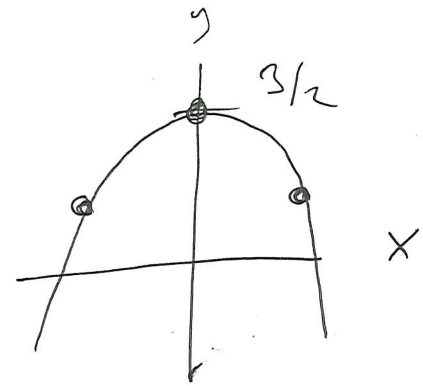
② $2y=\lambda$ ← can choose λ to make it true for any y

③ $y+0^2-1.5=0$

$y=1.5$

Point to Consider:

$(0, 3/2)$



Compare:

$f(x,y)=x^2+y^2$

$f(0,3/2)=0+\frac{9}{4}=\frac{9}{4}$

$f(1,1/2)=1+\frac{1}{4}=5/4$

$f(-1,1/2)=1+\frac{1}{4}=5/4$

Closest points:

$(\pm 1, 1/2)$

$(\text{Distance})^2 = 5/4$

Min distance: $\frac{\sqrt{5}}{2}$