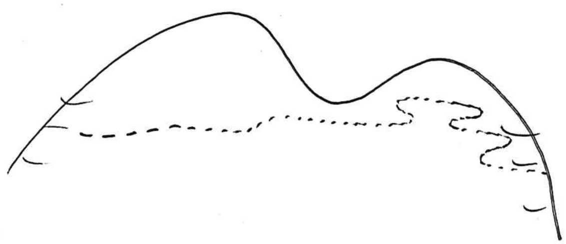
Problem #1

More investment leads to more returns, but you only have \$1000 to invest.

If $f(x,y)$ is your expected profit from investing \$$x$ in Space-X and \$$y$ in Yelp, optimize $f(x,y)$ subject to the constraint $x+y=1000$.

Problem #2



The height of a mountain at GPS coordinates (x,y) is given by the function $f(x,y)$.

A trail only traces a path along part of the mountain, say those positions (x,y) with $g(x,y)=0$.

What is the highest point reached by the trail?

Connection

Although the context is different, the questions are fundamentally asking the same thing.

Ballpark theorem:

Let f be a differentiable function in a region of $\mathbb{R}^2$ that contans a smooth curve C given by $g(x,y)=0$. Assume f has a local extreme value on C at point (a,b).

Then $\nabla f(a,b)$ is orthogonal to the line tangent to C at (a,b).

trail same direction as level curve

Assuming $\nabla g(a,b) \neq 0$, it follows there exist a real number λ (called a Lagrange multiplier) such that: $\boxed{\nabla f(a,b) = \lambda \nabla g(a,b)}$ $\otimes$ Takeaway

f: "objective function" ← want to maximize or minimize

$g(x,y)=0$ "constraint" defines "trail" (domain we care about)

Method:

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$$
$$\nabla g(a,b) = \langle g_x(a,b), g_y(a,b) \rangle$$

$$\nabla f(a,b) = \lambda \nabla g(a,b)$$

$$\Rightarrow \langle \underline{f_x(a,b)}, \overline{f_y(a,b)} \rangle = \langle \underline{\lambda \cdot g_x(a,b)}, \overline{\lambda \cdot g_y(a,b)} \rangle$$

① $f_x(a,b) = \lambda \cdot g_x(a,b)$
② $f_y(a,b) = \lambda g_y(a,b)$ (same λ, a, b)

Points that could be max/mins:

① $f_x(a,b) = \lambda g_x(a,b)$
② $f_y(a,b) = \lambda g_y(a,b)$ } (same λ) : my trail is flat for this instant

never use λ: just need it to exist

③ $g(a,b) = 0$ } need to be on curve.

Find all; compare.

ex Height of roller coaster: at point (x,y) $f(x,y) = xy + 14$

$g(x,y) = x^2 + y^2 - 18 = 0$

Only exists: $x^2 + y^2 = 18$

Find highest / lowest points.

Method of Lagrange Multipliers:

$$\boxed{\begin{array}{ll} f_x = y & g_x = 2x \\ f_y = x & g_y = 2y \end{array}}$$

$$\begin{cases} ① \ y = \lambda \cdot 2x \longrightarrow \lambda = \dfrac{y}{2x} \quad \text{OR} \quad x = 0 \\ ② \ x = \lambda \cdot 2y \\ ③ \ x^2 + y^2 - 18 = 0 \end{cases}$$

$y = \lambda \cdot 2x$

$0 = \lambda \cdot 0$

Usually:

$ax = bx$

$\Rightarrow a = b$

EXCEPT:

$2 \cdot 0 = 5 \cdot 0$

$2 \neq 5$

$\boxed{\text{If } \lambda = \frac{y}{2x}}$

① true

② $x = \left(\frac{y}{2x}\right) \cdot 2y$

λ

$x = \frac{2y^2}{2x}$

$2x^2 = 2y^2$ So $\boxed{x = \pm y}$

③ $x^2 + y^2 - 18 = 0$

$x^2 + x^2 - 18 = 0$

$2x^2 = 18$

$x = \pm 3$

$\boxed{\text{Points to Consider}}$

$(3, 3)$ $(-3, 3)$

$(3, -3)$ $(-3, -3)$

$\boxed{\text{If } x = 0}$

① $y = \lambda \cdot 2x$

$\Rightarrow y = \lambda \cdot 0$

$\Rightarrow y = 0$

$(0, 0)$

② $x = \lambda \cdot 2y$

$0 = \lambda \cdot 0$

true

③ $x^2 + y^2 - 18 = 0$

$0^2 + 0^2 - 18 \neq 0$

FALSE

No points to consider from this case

$f(x,y) = xy + 14$

$f(3,3) = 9+14 = 23$

$f(3,-3) = -9+14 = 5$

$f(-3,3) = -9+14 = 5$

$f(-3,-3) = 9+14 = 23$

23: highest, @ $(3,3)$, $(-3,-3)$ || 5: lowest, $(3,-3)$, $(-3,3)$

ex Find the largest x-value on the ellipse

$$x^2 - 2xy + 5y^2 = 1$$

Want to find max : x

$$f(x,y) = x$$

"objective function"

$f_x = 1$

$f_y = 0$

Constraint : $g(x,y) = x^2 - 2xy + 5y^2 - 1 = 0$

$g_x = 2x - 2y$

$g_y = -2x + 10y$

① $1 = \lambda(2x - 2y)$

② $0 = \lambda(-2x + 10y)$

③ $x^2 - 2xy + 5y^2 - 1 = 0$

$\rightarrow \boxed{\lambda = \frac{1}{2x - 2y}}$ $\boxed{\text{OR}}$ $\boxed{2x - 2y = 0}$

$\boxed{\text{If } \lambda = \frac{1}{2x-2y}}$

ASSUMING: $2x-2y \neq 0$

① true

②: $0 = \lambda(-2x+10y)$ $\quad \lambda \neq 0$

$0 = -2x+10y$

$2x = 10y$

$\boxed{x = 5y}$

Points to consider

$$\boxed{\left(\frac{5}{\sqrt{20}}, \frac{1}{\sqrt{20}}\right) \quad \left(\frac{-5}{\sqrt{20}}, \frac{-1}{\sqrt{20}}\right)}$$

③ $x^2 - 2xy + 5y^2 - 1 = 0$

$(\underbrace{5y}_{x})^2 - 2(\underbrace{5y}_{x})y + 5y^2 - 1 = 0$

$25y^2 - 10y^2 + 5y^2 = 1$

$20y^2 = 1$

$y^2 = \frac{1}{20}$

$y = \pm \frac{1}{\sqrt{20}}$

$\boxed{\text{If } 2x-2y=0}$ ① $1 = \lambda \underbrace{(2x-2y)}_{0}$

FALSE

No points to consider

$f\left(\frac{5}{\sqrt{20}}, \frac{1}{\sqrt{20}}\right): \frac{5}{\sqrt{20}}$

$f\left(\frac{-5}{\sqrt{20}}, \frac{-1}{\sqrt{20}}\right): \frac{-5}{\sqrt{20}}$

Lgst x-value: $\frac{5}{\sqrt{20}}$