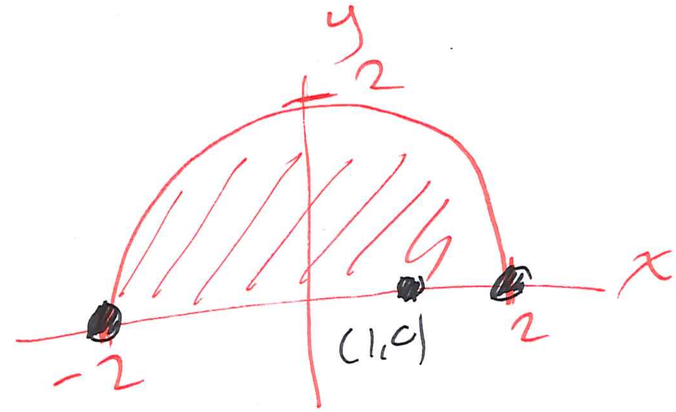
(ex) Find all absolute extrema of

$$f(x,y) = x^2 + y^2 - 2x + 2$$

over the region bounded by

$$\underbrace{x^2 + y^2 \le 4}_{\text{circle, radius 2}} \quad \text{and} \quad \underbrace{y \ge 0}$$

circle, radius 2
ctr: (0,0)



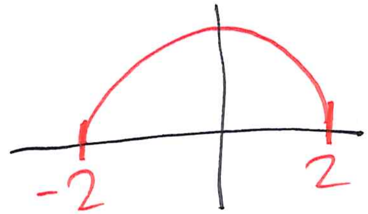
Abs extrema occur at CPs + on boundary

Find CPs: $f_x = 2x - 2 = 0 \longrightarrow x = 1$ only CP: (1,0)

$f_y = 2y = 0 \longrightarrow y = 0$ (on bdry)

Boundary

Piece 1: $x^2+y^2=4$



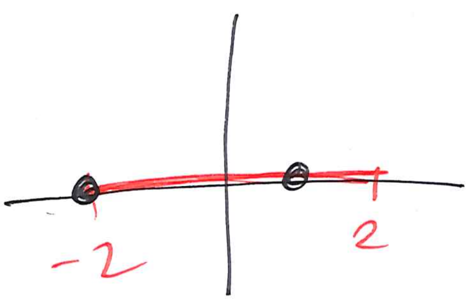
$$f(x,y) = \underbrace{x^2+y^2}-2x+2$$
$$= 4-2x+2$$
$$= 6-2x$$

big for small x
small for big x

From picture:
$-2 \leq x \leq 2$

If $x=-2$, $y=0$, $f(-2,0) = 6-2(-2) = 10$
If $x=2$, $y=0$, $f(2,0) = 6-2(2) = 2$

Piece 2: $y=0$



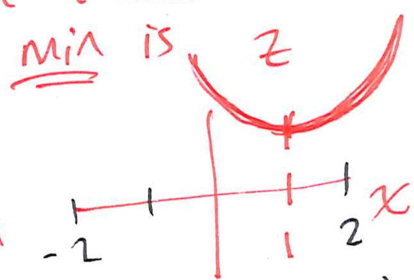
$$f(x,y) = x^2+y^2-2x+2$$
$$= x^2-2x+2$$

Parabola
Pointing up
We want to know
where its min is

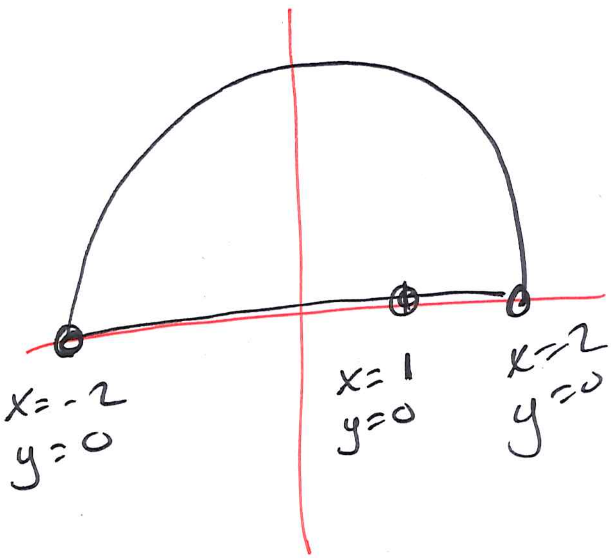
2 ways

① Find CP of
$$g(x) = x^2-2x+2$$
$$g'(x) = 2x-2 \quad x=1 \quad \text{min}$$

② Complete the square:
$$x^2-2x+2 =$$
$$x^2-2x+1+1 =$$
$$= (x-1)^2+1$$



MIN: $x=1$ $(y=0)$
MAX: $x=-2$ $(y=0)$



Points to consider

$f(-2,0) = 4+0-2(-2)+2 = 10$ Abs max

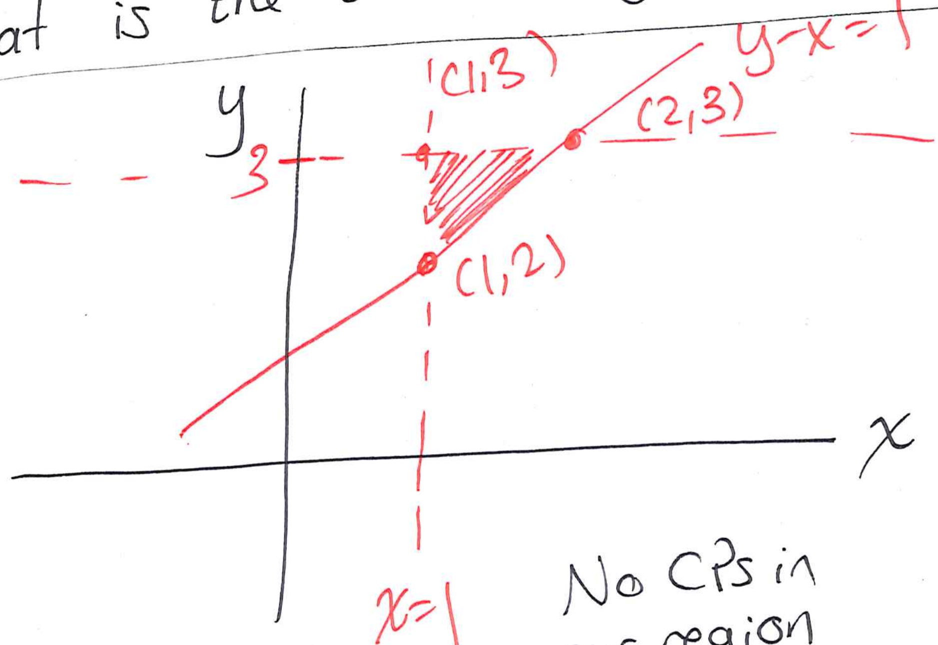
$f(1,0) = 1+0-2+2 = 1$ Abs min

$f(2,0) = 4+0-4+2 = 2$ (nothing in particular) ex To manufacture a gear with inner radius x and outer radius y costs $x(x-y)^2$ dollars.

For manufacturing, we need:

1. $y - x \geq 1$ (not too thin)
2. $x \geq 1$ (big enough hole)
3. $y \leq 3$ (not too big)

What is the cheapest gear we can make?



No CP's in our region

$$f_x = x \cdot 2(x-y) + (x-y)^2$$
$$= (x-y)(2x) + (x-y)(x-y)$$
$$= (x-y)(2x+x-y)$$
$$= (x-y)(3x-y) = 0$$

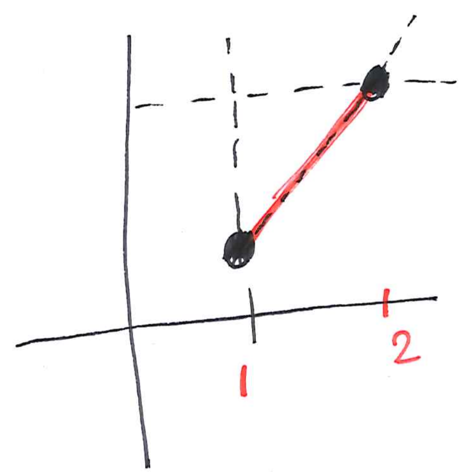
$$f_y = -2x(x-y) = 0$$

$\underbrace{x=0}_{\text{NO! } x \geq 1}$ or $\underbrace{y=x}_{\text{No! } y-x=0 \text{ Need } y-x \geq 1}$

Check Bdry:

$\boxed{\text{Piece 1}}$ $\quad y - x = 1$

cost: $x\underbrace{(x-y)}_{-1}{}^2 = x$



(from picture)

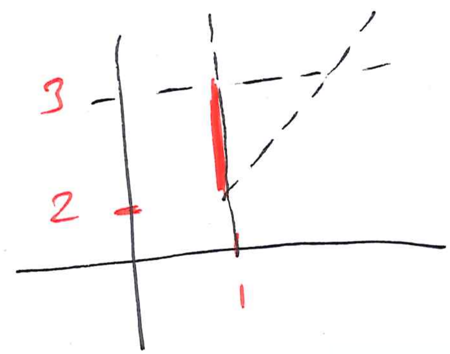
$1 \le x \le 2$

cost: min along piece 1

$c(1,2) = 1$

$c(2,3) = 2$

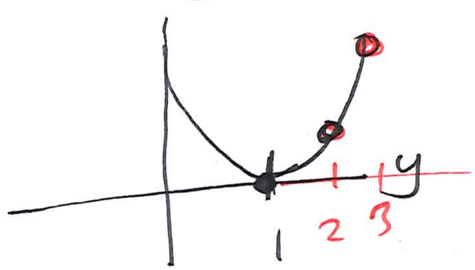
$\boxed{\text{Piece 2}}$ $\quad x = 1$



(from picture)

cost: $x(x-y)^2 = 1(1-y)^2$

$= (1-y)^2$



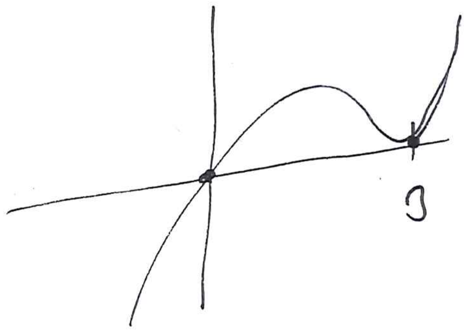
$\underline{2 \le y \le 3}$

Min: $c(1,2) = (1-2)^2 = 1$

Max: $c(1,3) = (1-3)^2 = 4$

$\boxed{\text{Piece 3}}$ $y = 3$

$$C(x,y) = x(x-y)^2$$
$$= x(x-3)^2$$



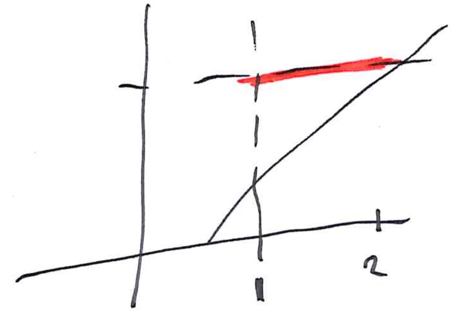
No CPs in REGION

Piece 1: cheapest $x=1,\ y=2 \quad C=1$

Piece 2: cheapest $x=1 \quad y=2 \quad c=1$

Piece 3: cheapest $x=2,\ y=3,\ C=2$

*



Want to know:

min of $x(x-3)^2$ when $1 \le x \le 2$

Calc 1 problem

Extrema of $y=f(x)$ occur @ CPs and endpoints.

$$g(x) = x(x-3)^2 = x(x^2-6x+9) = x^3-6x^2+9x$$

$$g'(x) = 3x^2-12x+9$$
$$= 3(x^2-4x+3)$$
$$= 3(x-1)(x-3)$$

CPs: $x=1 \quad y=3$

$\cancel{x=3}$ not in our region

EPs: $g(1) = 1(-2)^2 = 4$

$g(2) = 2(2-1)^2 = 2$ ← cheapest when $y=3$

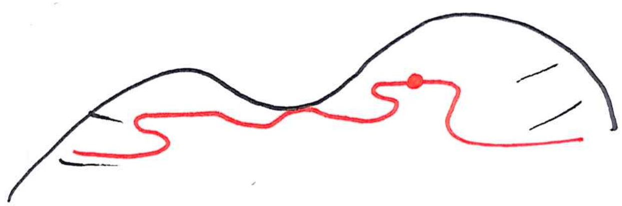
CP: $g(1) = 1(1-3)^2 = \cancel{0}\ 4$

Should read: g(1)=1(1-3)^2=4

To find extrema: (over R)

1. Find CPs
2. Find max/min on bdry
   (sometimes need, Calc I)
3. Compare

12.9 Constrained optimization

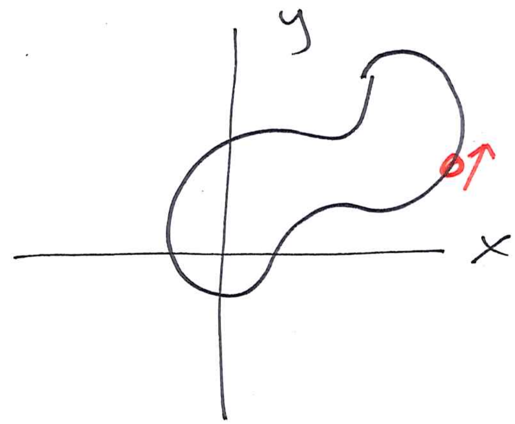


Where is highest pt of trail?

At max/min along trail:
constraint fcn (trail)
will be parallel to level
curve of surface.

We need a tool for describing direction of a level curve

and direction of a closed curve in $\mathbb{R}^2$



(Intuition behind Lagrange Multipliers)

Tool: gradient

The gradient of $z = f(x,y)$ is the vector

$$\langle f_x , f_y \rangle = \nabla f(x,y)$$

At a point (a,b), the gradient $\langle f_x(a,b), f_y(a,b)\rangle$ is orthogonal to the level curve, and points in direction of greatest increase of the function.

eg If $\nabla f(2,5) = \langle 1, 17 \rangle$

By walking in direction $\langle 1, 17 \rangle$

you'll go uphill as steeply as possible