- Webwork 2, Question 5 had a missing file.
  It is marked correct for everyone.

- You should attend the section you're registered in.
  If you are sitting here you should be registered in Section 208.

From last time:

We call a point (a,b) a <u>critical point</u> of a function f if:

- $f_x(a,b) = f_y(a,b) = 0$
- AND/OR $f_x(a,b)$ does not exist
- AND/OR $f_y(a,b)$ does not exist.

Every local extremum (max/min) occurs at a critical point, but not every critical point is the location of a local extremum.

ex) $Z = 16 - 4x^2 - y^2$

$Z_x = -8x = 0 \Rightarrow x = 0$

$Z_y = -2y = 0 \Rightarrow y = 0$

Only CP: $(0,0)$

ex) $Z = x - y^2$ No CPs

$Z_x = 1$ (<u>never</u> zero)

ex) Find all CPs of $f(x,y) = x^2 + xy + y^2 + 6x + 9$

$f_x = 2x + y + 6 = 0$

$f_y = x + 2y = 0 \longrightarrow x = -2y$ at any CP

If $2x + y + 6 = 0$, then

$2(\underbrace{-2y}_{x}) + y + 6 = 0$: $-3y + 6 = 0$, so $\boxed{y = 2}$

$x = -2y$, so $\boxed{x = -4}$

$\boxed{\text{Only CP: } (-4, 2)}$

(ex) Find all CP of $f(x,y) = x^3 + y^2 + xy + 2x + y$

$f_x = 2x + y + 2 = 0 \rightarrow y = -2 - 2x$

$f_y = 2y + x + 1 = 0$

$2(\underbrace{-2-2x}_{y}) + x + 1 = 0$

$-4 - 4x + x + 1 = 0$

$-3x - 3 = 0$

$\boxed{x = -1}$

$y = -2 - 2(-1)$

$\boxed{y = 0}$

$\boxed{CP: (-1, 0)}$

Check:

$f_x(-1,0) = 2(-1) + 0 + 2 = 0$

$f_y(-1,0) = 2 \cdot 0 - 1 + 1 = 0$

ex) Find all CPs of $f(x,y) = x^2 + y^3 + xy + 5x$

$f_x = 2x + y + 5 = 0$

$f_y = 3y^2 + x = 0 \longrightarrow x = -3y^2$

So: $\underbrace{2(-3y^2)}_{x} + y + 5 = 0$

$-6y^2 + y + 5 = 0$ quadric equation

$6y^2 - y - 5 = 0$

$(y-1)(6y+5) = 0$

So: $y = 1$, $x = -3$

or $y = -5/6$, $x = -3\left(\frac{25}{36}\right)$

$= \frac{-25}{12}$

$$\boxed{\begin{aligned} \text{CPs: } & (-3, 1) \\ & \left(-3 \cdot \tfrac{25}{36}, \tfrac{-5}{6}\right) \end{aligned}}$$

Classifying CPs using 2^{nd} Derivative Test

If a point (x,y) has:

$$\left.\begin{array}{l} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{array}\right\} \text{CP}$$

and:

$f(a,b) > f(x,y)$ for some nearby (a,b) ← not max

$f(c,d) < f(x,y)$ " " " (c,d) ← not min

we call (x,y) a <u>saddle point</u>.

2^{nd} Derivative Test (Theorem 12.14, p 941)

Suppose that the second partial derivatives of f are continuous near (a,b), ← CP

where $f_x(a,b) = f_y(a,b) = 0$.

Let $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - f_{xy}^2(x,y)$.

① If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then f has a local min at (a,b)

② If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has a local max at (a,b)

③ If $D(a,b) < 0$, then f has a saddle point at (a,b)

④ If $D(a,b) = 0$, the test is inconclusive — use a different method.

(won't prove it)

ex $f(x,y) = x^4 + 2y^2 - 4x$

Find all local extrema

① Find CPs ② Classify using 2^{nd} deriv test

① $f_x = 4x^3 - 4 = 0 \longrightarrow 4x^3 - 4 = 0 \quad x^3 - 1 = 0 \quad x^3 = 1 \quad \boxed{x=1}$

$f_y = 4y = 0 \longrightarrow \boxed{y=0}$ Only CP: $\boxed{(1,0)}$

② $D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - f_{xy}^2(x,y)$

$f_{xx} = 12x^2 \qquad f_{xy} = 0 \qquad$ So: $D(x,y) = 48x^2 - 0^2 = 48x^2$

$f_{yy} = 4$

$D(1,0) = 48 \cdot 1 > 0$; $f_{xx}(1,0) = 12 > 0$

Local min @ $(1,0)$

$f(1,0) = 1 + 0 - 4 = -3$

ex Find and classify all CPs of

$$f(x,y) = x^2 + xy^2 - 2x + 1$$

$f_x = 2x + y^2 - 2 = 0$

$f_y = 2xy = 0 \longrightarrow x = 0$ AND/OR $y = 0$

If $x = 0$:

$f_x = 0$

$\Rightarrow y^2 - 2 = 0$

$y = \pm\sqrt{2}$

CPs: $(0, \sqrt{2})$

$(0, -\sqrt{2})$

If $y = 0$:

$f_x = 0$

$2x - 2 = 0$

$x = 1$

CP: $(1, 0)$

(ex) Find and classify all CPs of

$$f(x,y) = (x+2)^2 + (3y-6)^2$$

$f_x = 2(x+2) = 0 \longrightarrow x = -2$ CP: $(-2, 2)$

$f_y = 2(3y-6)\cdot 3 = 0 \longrightarrow y = 2$

$f_{xx} = 2$ $f_{xy} = 0$

$f_{yy} = 6\cdot 3$

$D(x,y) = 2\cdot 18 - 0^2 = 36 > 0$

$D(-2,2) > 0$

$f_{xx}(-2,2) = 2 > 0$

min @ $(-2, 2)$

ex $f(x,y) = (x+2)^2 + (3y-6)^4$

(skipping steps)

$(-2, 2)$ CP

$D(-2,2) = 0 \longrightarrow$ 2nd deriv inconclusive

$(x+2)^2 \geq 0$
$(3y-6)^4 \geq 0$ so $f(x,y) \geq 0+0=0$

$f(-2,2) = 0^2 + 0^4 = 0$

So: $(-2,2)$ is location of local <u>minimum</u>

<u>Every</u> value of $f(x,y)$ is at least this big (or bigger)

$f_{xx} = 2 \qquad f_{xy} = 2y$

$f_{yy} = 2x$

$D(x,y) = 2 \cdot 2x - (2y)^2 = 4x - 4y^2$

- $D(1,0) = 4 \cdot 1 - 4 \cdot 0 = 4 > 0$

  $f_{xx}(1,0) = 2 > 0$

- $D(0, \pm\sqrt{2}) = 4 \cdot 0 - 4 \cdot 2 < 0$

Local min @ $(1,0)$

$f(1,0) = 1 - 2 + 1 = 0$

Saddle points @

$(0, -\sqrt{2})$ and $(0, +\sqrt{2})$