Friendly Reminders:

- Info about Wednesday's (short) quiz
  is online
- If you are sitting in this section,
  you need to be registered in this section.
  (If you're registered in another section, please
  attend it.)

Classifying CPs

Local Max
Local Min
Saddle Point

2nd Derivative Test

Suppose that the second-order partial derivatives of f are continuous around (a,b) where

CPs [$f_x(a,b) = f_y(a,b) = 0$.

Let $D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - f_{xy}^2(x,y)$

1. If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$ $\Rightarrow$ f has local max at (a,b)
2. If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$ $\Rightarrow$ f has local min at (a,b)
3. If $D(a,b) < 0$, then f has saddle point at (a,b)
4. If $D(a,b) = 0$, test inconclusive (need to try another way)

we won't prove

ex

$$f(x,y) = x^4 + 2y^2 - 4xy$$

Find all local extrema

① Find all CPs ✓ ② Classify using 2^{nd} deriv test

① $f_x = 4x^3 - 4y = 0$

$f_y = 4y - 4x = 0 \rightarrow 4y = 4x$

So $\boxed{y = x}$

If (x,y) is a CP then $x = y$

Also: $4x^3 = 4y = 4x$

$4x^3 = 4x$

$x^3 = x$

$x^3 - x = 0$

$x(x^2 - 1) = 0$

$x(x+1)(x-1) = 0$

$x = 0$

$x = -1$

$x = 1$

$$\boxed{\begin{matrix} (0,0) \\ (-1,-1) \\ (1,1) \end{matrix}}$$

CPs

(ex) Find and classify all CPs of:

$$f(x,y) = x^2 - 2x - y^2 - 4y - 4$$

$f_x = 2x - 2 = 0$ $\quad 2x = 2 \quad \boxed{x = 1}$ $\quad \boxed{(1, -2)}$

$f_y = -2y - 4 = 0$ $\quad 2y = -4 \quad \boxed{y = -2}$ $\quad$ only CP

$f_{xx} = 2 \qquad f_{xy} = 0 \qquad D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - f_{xy}^2(x,y)$

$f_{yy} = -2 \qquad\qquad = 2(-2) - 0^2 = -4 < 0$

2nd Deriv Test: $\boxed{(1,-2) \text{ saddle pt}}$

(ex) Find all local extrema of

$$f(x,y) = \underbrace{(x+2)^2}_{\geq 0} + \underbrace{(3y-6)^4}_{\geq 0} \geq 0$$

$f_x = 2(x+2) = 0$ $\boxed{x=-2}$

$f_y = 4(3y-6)^3 \cdot 3 = 0$ $\boxed{y=2}$

Only CP: $(-2,2)$

$f_{xx} = 2$ $f_{xy} = 0$

$f_{yy} = 12 \cdot 3(3y-6)^2 \cdot 3$

$= 12 \cdot 9(3y-6)^2$

$D(x,y) = 2 \cdot 12 \cdot 9(3y-6)^2 - 0^2$ 2^{nd} Deriv Test inconclusive

$D(-2,2) = 2 \cdot 12 \cdot 9 \cdot 0 = 0$

$f(-2,2) = 0^2 + 0^2 = \boxed{0}$

$\left.\begin{array}{l}(x+2)^2 \geq 0 \\ (3y-6)^4 \geq 0\end{array}\right\} f(x,y) \geq 0$

CP: $\boxed{\text{minimum}}$

② $D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - f_{xy}^2(x,y)$

$f_{xx} = 12x^2$ $\quad f_{xy} = -4$

$f_{yy} = 4$

$\boxed{D(x,y) = 48x^2 - 16}$

$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(4x^3 - 4y) = 0 - 4$

- $D(0,0) = -16 < 0$
  So: @ $(0,0)$ saddle point
- $D(1,1) = 48 - 16 > 0$
  $f_{xx}(1,1) = 12(1) > 0$
  So: @ $(1,1)$ local min
  $f(1,1) = 1 + 2 - 4 = -1$
- $D(-1,-1) = 48 - 16 > 0$
  $f_{xx}(-1,-1) = 12 > 0$
  So: @ $(-1,-1)$ local min
  $f(-1,-1) = 1 + 2 - 4 = -1$

Classify the origin

- $f(x,y) = \cos(xy)$

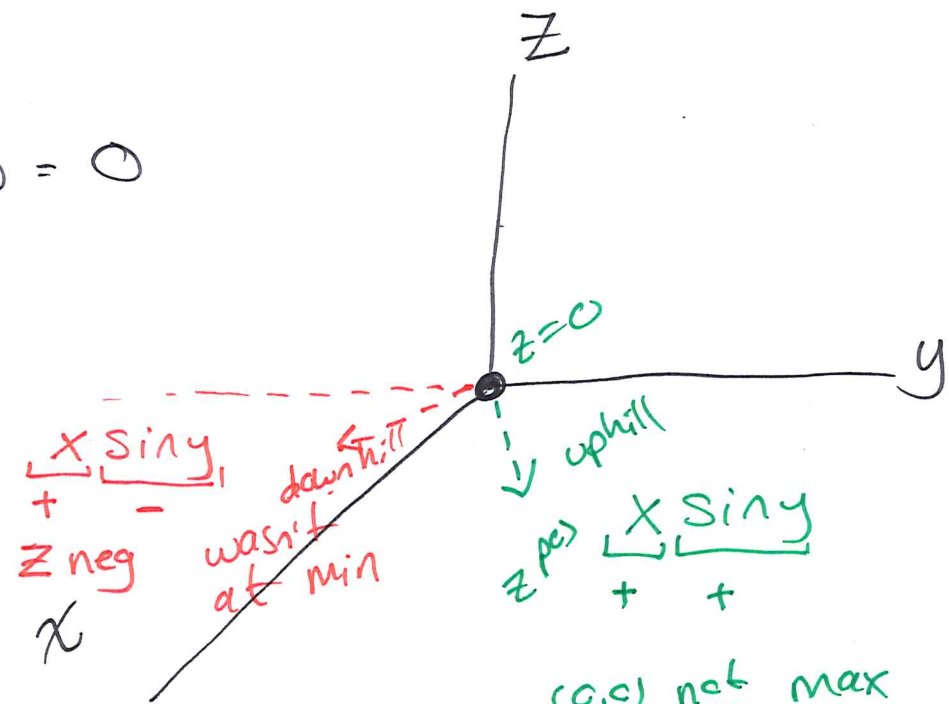
  $f(0,0) = \cos 0 = 1$

  MAX at $(0,0)$

- $f(x,y) = x \sin y$

  $f(0,0) = 0 \cdot \sin 0 = 0$

  $\boxed{\text{saddle point}}$



(ex) Let $f(x,y) = \sin x + \cos y$.
Classify $(0,0)$, $(\pi/2, 0)$, and $(\pi/2, \pi)$

(not done in class)

$f_x = \cos x$

$f_y = -\sin y$

$f_{xx} = -\sin x$

$f_{yy} = -\cos y$

$f_{xy} = 0$

$D(x,y) = \sin x \cos y$

$f_x(0,0) = 1$, so $(0,0)$ is not a critical point.

$f_x(\pi/2, 0) = \cos \pi/2 = 0$ and $f_y(\pi/2, 0) = \sin 0 = 0$, so $(\pi/2, 0)$ is a CP.

$f_x(\pi/2, \pi) = \cos \pi/2 = 0$ and $f_y(\pi/2, \pi) = \sin \pi = 0$, so $(\pi/2, \pi)$ is a CP.

$D(\pi/2, 0) = \sin(\pi/2)\cos(0) = 1 > 0$

$f_{xx}(\pi/2, 0) = -\sin(\pi/2) = -1 < 0$

So $(\pi/2, 0)$ is a local max

$D(\pi/2, \pi) = \sin(\pi/2)\cos(\pi) = 1(-1) = -1 < 0$

So $(\pi/2, \pi)$ is a saddle point