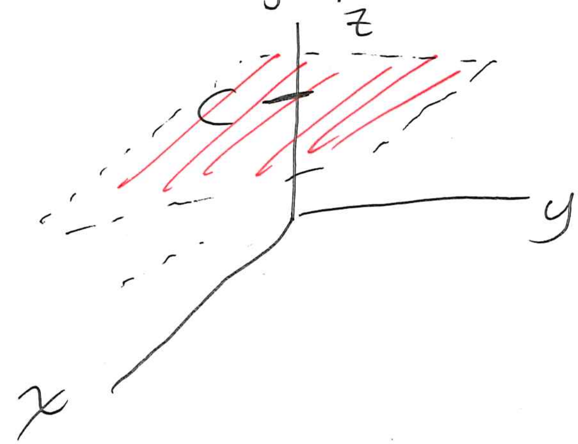
Information about the next quiz is on our section's website

I write things there because I want you to read them :)

Last time I should have defined <u>traces</u>

In sketching $z = f(x,y)$

If we set $z = \text{constant}$, we get a <u>level</u> <u>curve</u> (horizontal cross-section) : intersection of our surface with a plane parallel to the xy-plane



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Similarly, if we set $y = \text{constant}$ or $x = \text{constant}$ we get

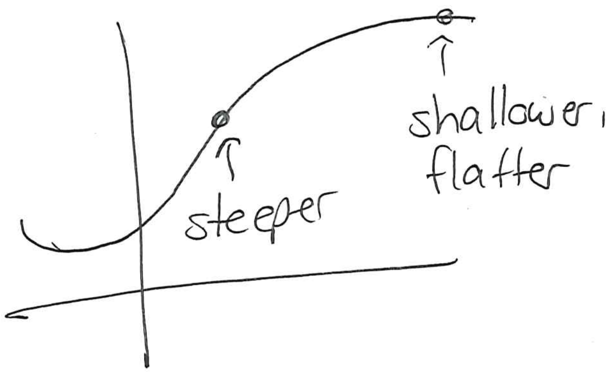
the intersection of our surface with a (vertical) plane parallel to: xz-plane, yz-plane

These, and level curves, are called <u>traces</u>.

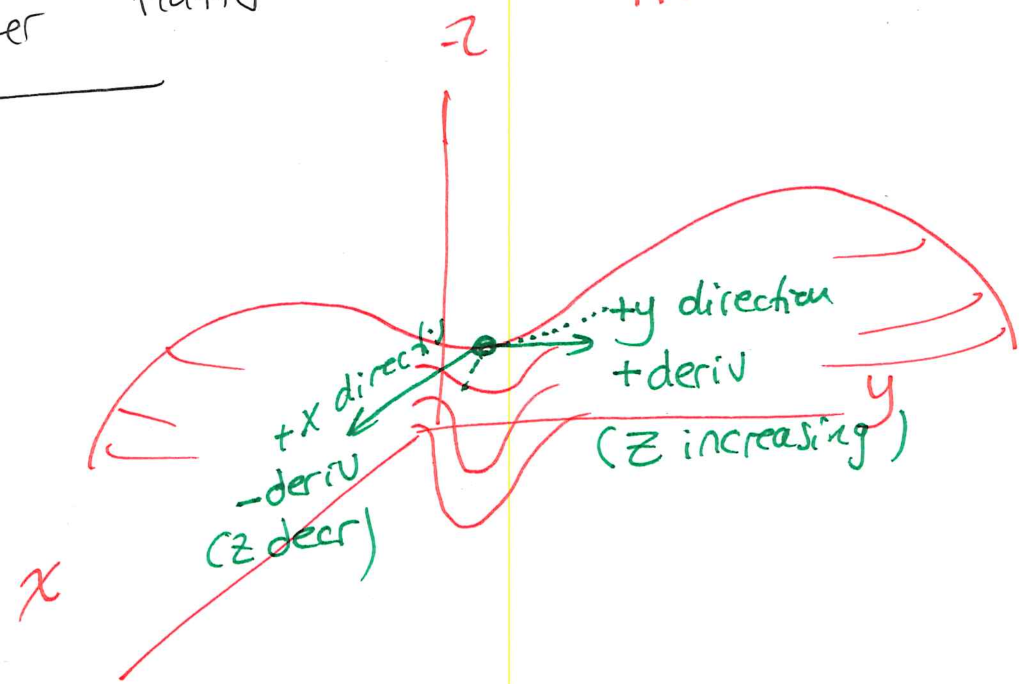
Idea: set one variable $\to$ constant
You get a trace

Ch 12.4 : partial derivatives

Remember in $\mathbb{R}^2$: derivative $\leftrightarrow$ slope

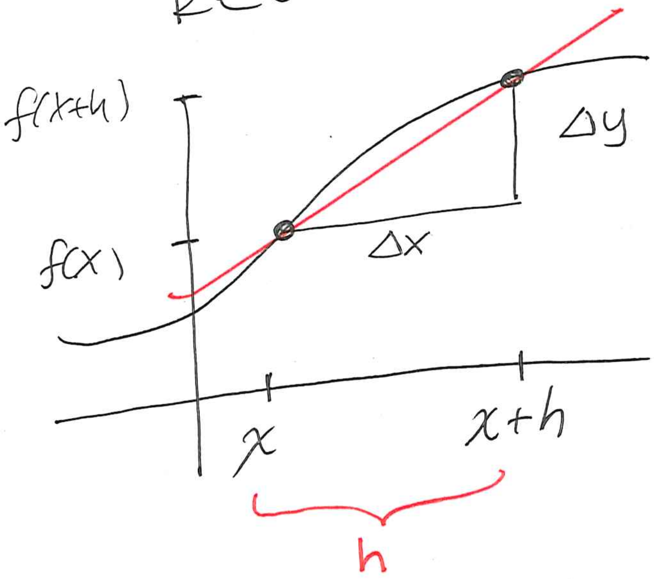


In $\mathbb{R}^3$ direction matters



Review: derivatives in $\mathbb{R}^2$

slope: $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x+h) - f(x)}{h}$ (avg rate of change)



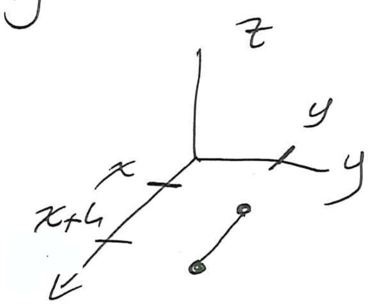
Derivative (instantaneous rate of change)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Generalize to $\mathbb{R}^3$: two partial derivatives

One kind: we move only in x-direction, y stays constant.

output: z (not y)



$$\frac{\Delta z}{\Delta x} = \frac{f(x+h, y) - f(x, y)}{h}$$

y is constant! So looks like

$\dfrac{g(x+h) - g(x)}{h}$ } looks like derivative in $\mathbb{R}^2$!

To find partial derivative of $z = f(x,y)$ with respect to x : treat x as variable, y constant

y : y - variable, x - constant

ex) $f(x,y) = 3xy^2 - 15x + \ln y$

$\underbrace{(3y^2)}_{\#} x$

Partial deriv wrt x :

$$\frac{\partial f}{\partial x} = f_x(x,y) = f_x = 3y^2 - 15 + 0 = \boxed{3y^2 - 15}$$

(no prime)

Partial deriv wrt y :

$\underbrace{(3x)}_{\#} y^2$

$$\frac{\partial f}{\partial y} = f_y(x,y) = f_y = (3x) \cdot 2y + 0 + \frac{1}{y}$$

$$= \boxed{6xy + \frac{1}{y}}$$

ex

$$f(x,y) = 2x^2y + 3y^3 - \sin x$$

$$f_x(x,y) = (2y)\cdot 2x + 0 - \cos x$$

$$= \boxed{4xy - \cos x}$$

$$f_y(x,y) = 2x^2 + 9y^2 + 0$$

$$= \boxed{2x^2 + 9y^2}$$

a - const

$$\frac{d}{dy}(ay) = a$$

$$\frac{d}{dx}(ax) = a$$

ex

$$f(x,y) = (2x)y^2 - (5x+3y)^3$$

$$f_x(x,y) = 2y^2 - 3\underbrace{(5x+3y)}^2 \cdot 5$$

$$= \boxed{2y^2 - 15(5x+3y)^2}$$

$$f_y(x,y) = (2x)2y - 3(5x+3y)^2 \cdot 3$$

$$= \boxed{4xy - 9(5x+3y)^2}$$

Implicit Differentiation

ex: $x \cos z + z \cos y = 0$

ex: $\frac{d}{dx}\left(\sin(x^3+1)\right) = \cos(x^3+1) \cdot 2x$

Find z_x and z_y

z_x : x - variable
y - constant
z - function of x

$$x \cos z + z \overbrace{\cos y}^{\text{constant}} = 0$$

$$(1)\cos z + \underbrace{x(-\sin z \cdot z_x)}_{\text{chain rule, } z:\text{ fcn of } x} + [\cos y] z_x = 0$$

$$z_x(-x\sin z + \cos y) + \cos z = 0$$

$$z_x(-x\sin z + \cos y) = -\cos z$$

$$z_x = \frac{-\cos z}{-x\sin z + \cos y} = \boxed{\frac{\cos z}{x \sin z - \cos y}}$$

$$\overbrace{x}^{\text{const}}\cos z + \underbrace{z}\,\underbrace{\cos y} = 0$$

z_y :
x-constant
y-variable
z-fcn of y

$$x(-\sin z \cdot z_y) + z(-\sin y) + \cos y \cdot z_y = 0$$

$$z_y(-x\sin z + \cos y) = z\sin y$$

$$\boxed{z_y = \frac{z\sin y}{\cos y - x\sin z}}$$

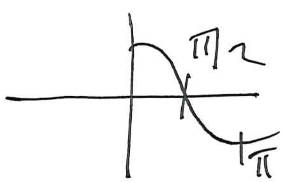
Interpretation

Notice: $x=0$, $y=2$, $z=0$

$$x\cos z + z\cos y = 0(1) + 0\cos 2 = 0$$

So: $(0,2,0)$ is on our surface.

$$z_x(0,2,0) = \frac{\cos 0}{0\sin(0) - \cos(2)} = \frac{1}{-\cos(2)} > 0$$



Standing at $(0,2,0)$ looking in $+x$ direction, the surface slopes <u>up</u> (its slope is $\frac{-1}{\cos 2}$)

Standing at $(0,2,0)$, looking in +y direction, the surface is flat

$$z_y = \frac{z \sin y}{\cos y - x \sin z} \qquad z_y(0,2,0) = \frac{0}{\cos 2 - 0} = 0$$

(ex) Your mark in Math 105 (out of 100) is:

$$M = 0.1\, W + 0.06\, Q + 0.34\, MT + 0.5\, F$$

$M_W = 0.1$

M: mark, W: webwork, Q: Quiz, MT: Midterms, F: Final

Explain: M_W $\quad M_Q$

Keeping other vars constant, M_W how a change in WW changes your final mark

$$M_W \doteq \frac{\Delta M}{\Delta W}$$

eg if $\Delta W = +1$

$$\frac{\Delta M}{1} = 0.1$$

$$\Delta M = 0.1$$

Higher-order partial derivatives

$$\begin{tikzcd}[column sep=small]
 & & & f(x,y) \arrow[dll] \arrow[dr] & & \\
 & f_x \arrow[dl] \arrow[d] & & & f_y \arrow[dl] \arrow[d] & \\
(f_x)_x & (f_x)_y & & (f_y)_x & (f_y)_y &
\end{tikzcd}$$

f_{xx} $\quad$ f_{xy} $\quad$ f_{yx} $\quad$ f_{yy}

1st: y
2nd: x

2 1st-order partial derivatives

4 2nd-order partial derivatives

ex $f(x,y) = x\sin y$

1st order: $f_x = \sin y$ $\qquad f_y = x\cos y$

2nd order: $f_{xx} = 0$ $\qquad f_{yx} = \cos y$

$f_{xy} = \cos y$ $\qquad f_{yy} = -x\sin y$

"mixed partial"

Clairaut's Theorem (Equality of Mixed Partial Derivatives)
Assume that f is defined on a open set D of $\mathbb{R}^2$, and that f_{xy} and f_{yx} are continuous throughout D.

Then: $f_{xy} = f_{yx}$

ex Is it possible a function $f(x,y)$ exists, defined on all real #s, with

$f_x = 3x$ and $f_y = 3x$?

Clairaut $\Rightarrow$ $f_{xy} = f_{yx}$

$f_{xy} = 0$ $f_{yx} = 3$

NOT POSSIBLE

ex Is it possible a function $f(x,y)$ exists, defined on all real #s, with $f_x = 3x^2+y$ and $f_y = x+2$?

Clairaut $\Rightarrow$ $f_{xy} = 1$ $f_{yx} = 1$

maybe

Actually: $\boxed{f(x,y) = x^3 + yx + 2y}$

$f_x = 3x^2+y$ $f_y = x+2$ ✓

(ex) Find a function (domain: $\mathbb{R}^2$) or show none exists:

(a) $f_x = y$ & $f_y = x$

$f(x,y) = xy$ (see before)

$f_{xy} = 1$
$f_{yx} = 1$

Clairaut's doesn't tell us anything

(b) $f_x = x$ & $f_y = y$

$f_{xy} = 0$
$f_{yx} = 0$

$\parallel$

Guess: $\frac{1}{2}x^2 + C$

Guess: $\frac{1}{2}y^2 + C$

$$\boxed{f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + C}$$

$f_x = x$ $f_y = y$ (check) ✓

Ch 12.8

$z = f(x,y)$

Can't (immediately) walk uphill

highest thing in its neighbourhood

flat in all directions:

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$$

"critical point"

A function f has a local $\begin{pmatrix} \text{max} \\ \text{min} \end{pmatrix}$

at (a,b) if $f(x,y) \begin{pmatrix} \leq \\ \geq \end{pmatrix} f(a,b)$

for all (x,y) in the domain of f in some open disk centred at (a,b)

If f has a local max or min at (a,b), and the partial derivatives f_x and f_y exist at (a,b), then $f_x(a,b) = f_y(a,b) = 0$

We call (a,b) a <u>critical point</u> of $f(x,y)$ if:

- $f_x(a,b) = f_y(a,b) = 0$

OR

- $f_x(a,b)$ DNE

OR

- $f_y(a,b)$ DNE

In this contect (and most mathematical contexts) "or" means "and/or"