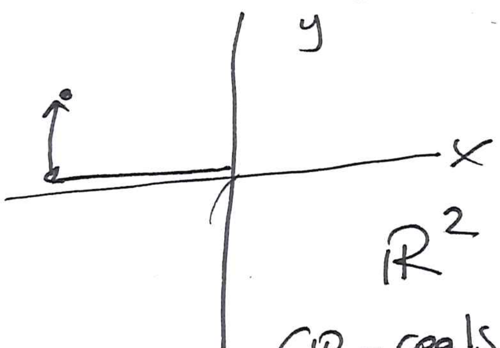
Math 105, Section 208

Chapter 11 : vectors

- vectors (gentle start)
  derivatives of multivariable functions (typically : Calc 3)
- continuous probability

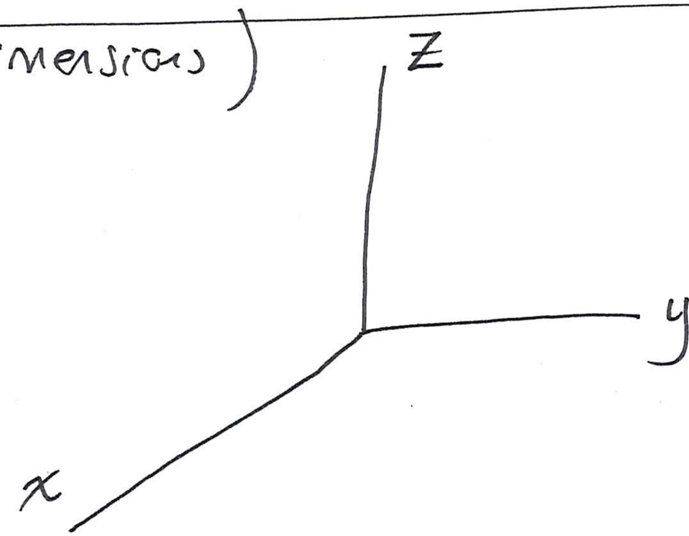
all Calc 2 $\left\{ \begin{array}{l} \text{integration} \\ \text{series} \end{array} \right.$

Sketching in $\mathbb{R}^3$ (3 dimensions)



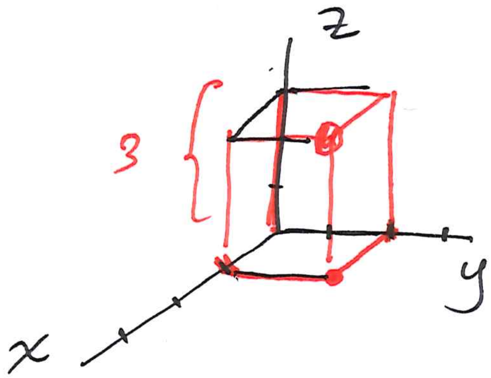
$\mathbb{R}^2$

($\mathbb{R}$ - reals)

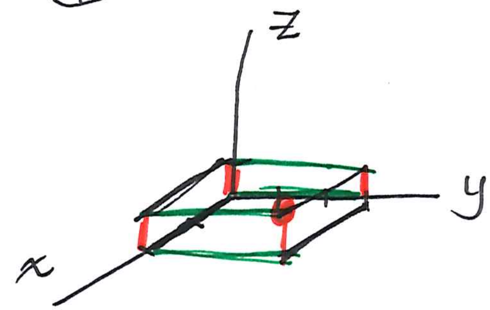


ex $(1, 2, 3)$

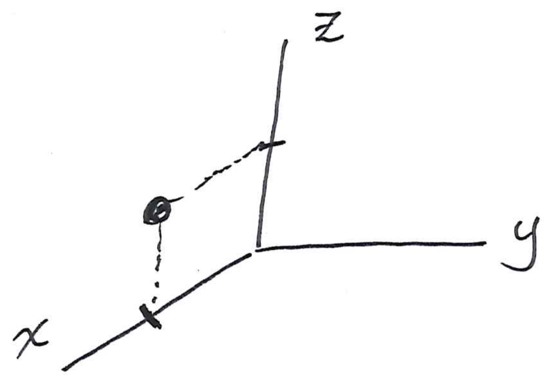
$x \quad y \quad z$



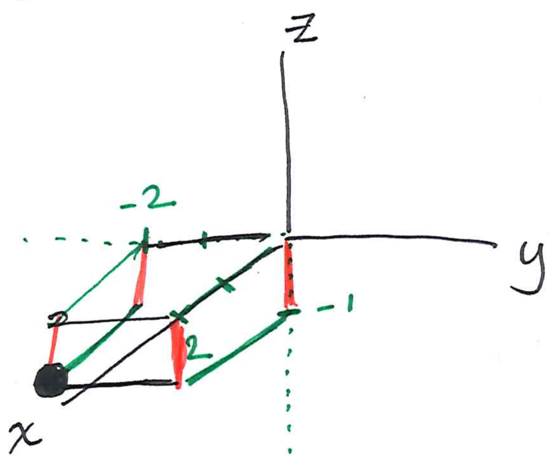
ex $(2, 3, 1)$



ex $(2, 0, 2)$



ex $(2, -2, -1)$



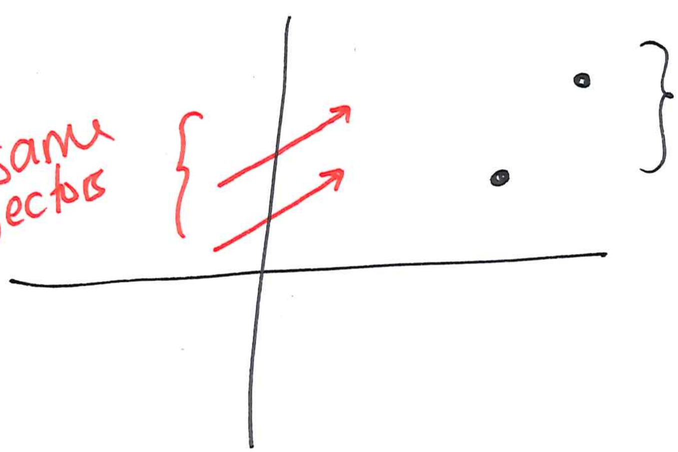
Vector: has magnitude and directions

usu: draw as arrows

hard push to right

soft push to right

Position not fixed



same vectors

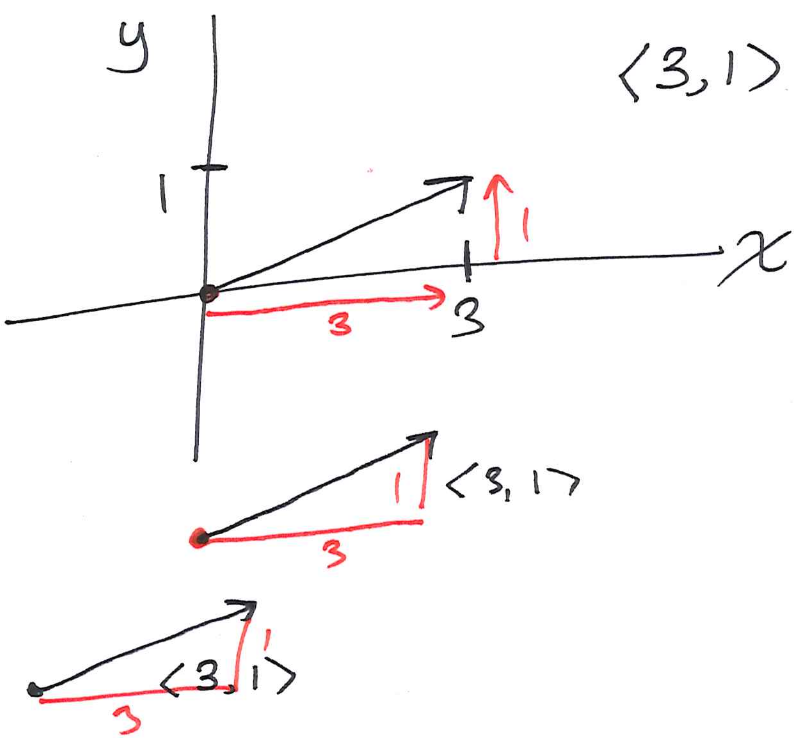
different coordinates

Algebraic operations on vectors (addition, etc)

Naming Vectors

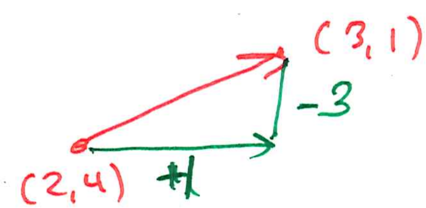
In a coordinate system, place the tail of the vector at the origin, then the coordinate of its head is the vector's name.

Name: instructions
To go from tail to head

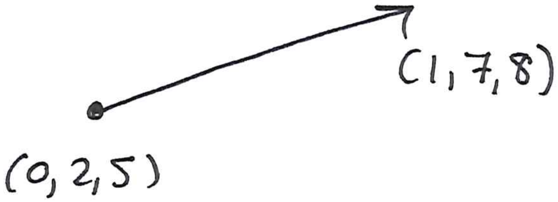


(ex) What is the name of a vector w/ tail at $(2, 4)$ and head at $(3, 1)$?

$\langle 1, -3\rangle$



ex



Vector name:

$\langle 1, 5, +3 \rangle$

$$\begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

x direction: 0 to 1 : +1

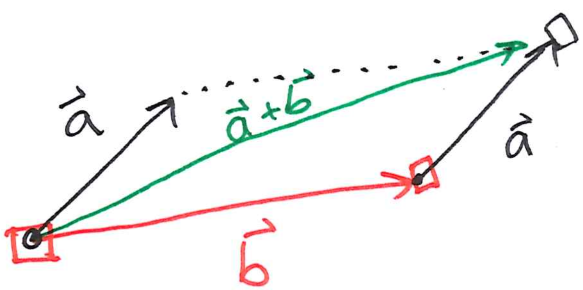
y direction: 2 to 7 : +5

z direction: 5 to 8 : +3

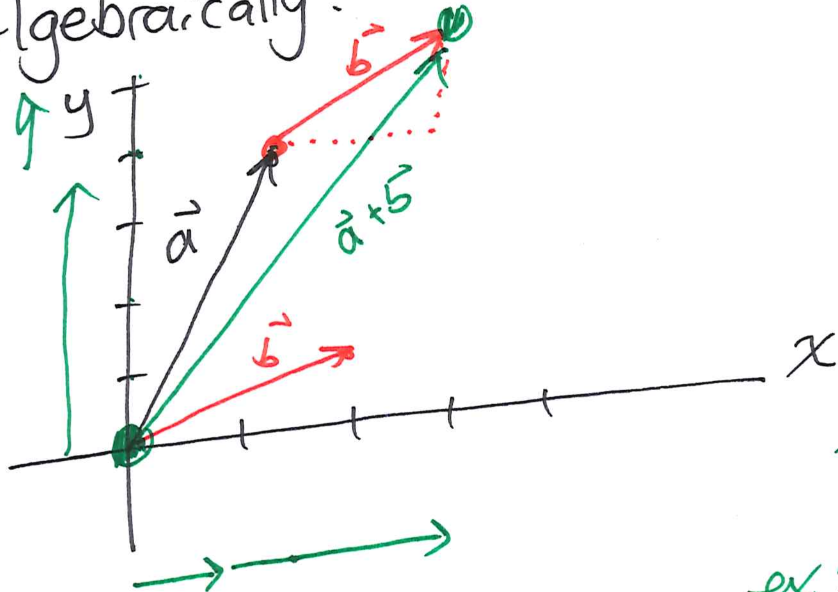
Adding Vectors

Geometrically:

$\vec{a} + \vec{b}$: put them head-to-tail



Algebraically:



$\vec{a} = \langle 1, 4 \rangle$

$\vec{b} = \langle 2, 1 \rangle$

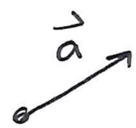
$\vec{a}+\vec{b} = \langle 3, \cancel{4}\ 5 \rangle$

ex:

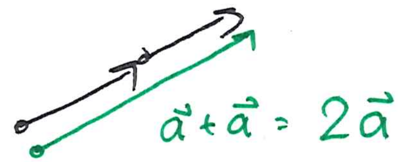
$$\begin{array}{r} \langle 1, 2, 17 \rangle \\ + \ \langle 8, -10, 3 \rangle \\ \hline \langle 9, -8, 20 \rangle \end{array}$$

Multiplying vectors with scalars

real number
$\mathbb{R}$
(not vector)

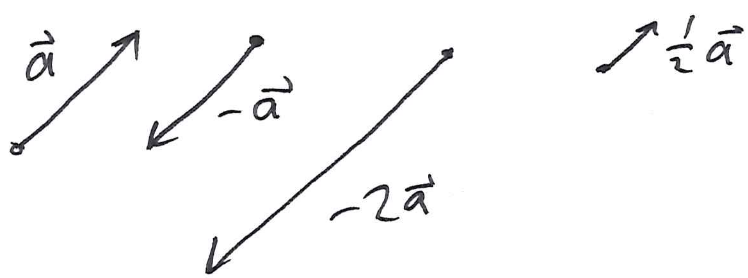


$\vec{a} + \vec{a}$:



When we multiply a vector by a scalar, we get a parallel vector, with scaled length.

If the scalar is negative, we reverse the direction.



Algebraically:

(ex) $2\langle 5, 9\rangle = \langle 10, 18\rangle$

(ex) $\frac{1}{4}\langle 16, 8, 32\rangle = \langle 4, 2, 8\rangle$

Fact: two vectors are parallel

$\Leftrightarrow$ one is a scalar multiple of other
$\in \mathbb{R}_{\neq 0}$,

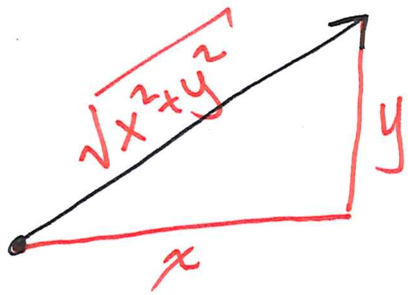
(ex) $\langle 2, 1, 5\rangle$ $\times 2$ parallel
$\langle 4, 2, 10\rangle$
$\langle 10, 5, 1\rangle$

not ∥

Which (if any) are parallel?

Length + Direction of vectors

Length



name : $\langle x, y\rangle$

In $\mathbb{R}^2$: $\vec{a} = \langle x, y\rangle$

$|\vec{a}| = \|\vec{a}\| = \sqrt{x^2+y^2}$

length

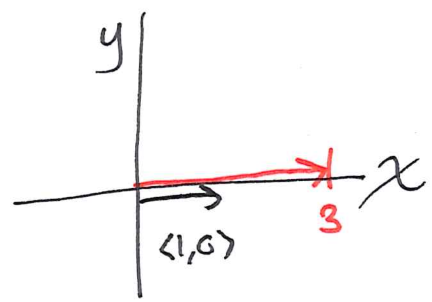
In $\mathbb{R}^3$: $\vec{a} = \langle x, y, z\rangle$

$|\vec{a}| = \sqrt{x^2+y^2+z^2}$

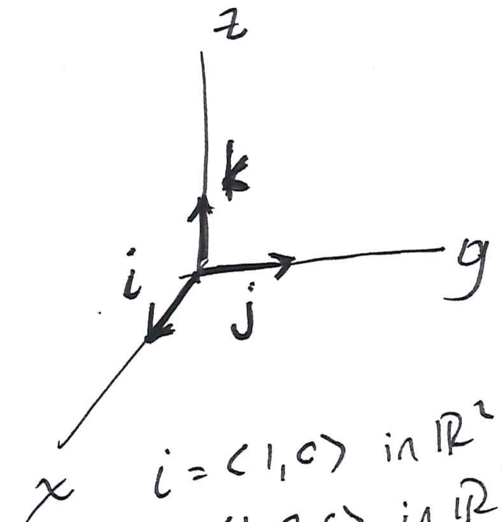
To describe a vector's direction, we find a vector in the same direction, with length 1.

<u>Unit vector</u> any vector of length 1.

ex Find a unit vector in direction of $\langle 3,0 \rangle$



$$\frac{1}{3}\langle 3,0 \rangle = \langle 1,0 \rangle$$



$i = \langle 1,0 \rangle$ in $\mathbb{R}^2$
$i = \langle 1,0,0 \rangle$ in $\mathbb{R}^3$
$j = \langle 0,1 \rangle$ in $\mathbb{R}^2$
$j = \langle 0,1,0 \rangle$ in $\mathbb{R}^3$
$k = \langle 0,0,1 \rangle$ in $\mathbb{R}^3$

ex Find a unit vector in direction of:

Idea: multiply by a positive scalar to make length be 1

- $\langle 1, 1 \rangle$ length: $\sqrt{1^2+1^2} = \sqrt{2}$
- $\langle 2, 3, 4 \rangle$ length: $\sqrt{4+9+16} = \sqrt{29}$
- $\langle a, b, c \rangle$ length: $\sqrt{a^2+b^2+c^2}$

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\langle 1,1\rangle = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Check: (positive) scalar mult of $\langle 1,1\rangle$, $\Rightarrow$ parallel

$$|\vec{u}_1| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = \sqrt{1} = 1 \quad \text{unit vector}$$

$$\vec{u}_2 = \frac{1}{\sqrt{29}}\langle 2,3,4\rangle = \left\langle \frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}} \right\rangle$$

$$\vec{u}_3 = \left\langle \frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}} \right\rangle$$

(as long as: $\langle a,b,c\rangle \neq \langle 0,0,0\rangle$)

Dot Product

another generalization of multiplication

Before: (scalar)(vector) $\to$ (vector)

Dot prod: (vector)(vector) $\to$ (scalar)

Define: $\langle a, b\rangle \cdot \langle x, y\rangle = \underbrace{ax + by}_{\text{scalar}}$

ex $\langle 2, 0, 15\rangle \cdot \langle 3, -10, -1\rangle = 6 + 0 + (-15) = -9$

why? Fact: If $\vec{a}$ and $\vec{b}$ are perpendicular
then $\vec{a} \cdot \vec{b} = 0$

Properties of Dot Product

Let $\vec{a}, \vec{b}$ be vectors ; s scalar

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ commut
- $\underset{\#}{s}\underbrace{(\vec{a} \cdot \vec{b})}_{\#} = \underbrace{\underbrace{(s\vec{a})}_{\text{vector}} \cdot \underbrace{\vec{b}}_{\text{vector}}}_{\text{scalar}}$ assoc
- $\underbrace{\underset{\text{vec}}{\vec{a}} \cdot \underbrace{(\vec{b}+\vec{c})}_{\text{vec}}}_{\text{scalar}} = \underbrace{\vec{a} \cdot \vec{b}}_{\text{sc}} + \underbrace{\vec{a} \cdot \vec{c}}_{\text{sc}}$ distrib

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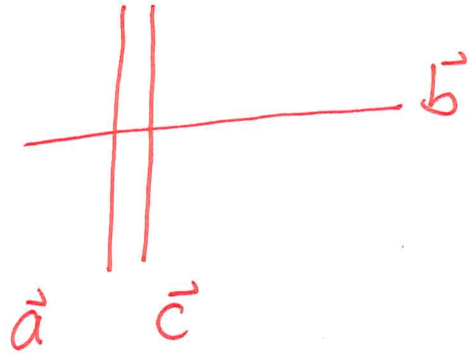
ex Let $\vec{a} = \langle 1, 0, 3 \rangle$

$\vec{b} = \langle 3, 0, -1 \rangle$

$\vec{c} = \langle -2, 0, -6 \rangle$

- which $\parallel$, $\perp$
- Calculate $(\vec{a}+\vec{b}) \cdot \vec{c}$

$-2\vec{a} = \vec{c}$



$\vec{a} \cdot \vec{b} = 3 + 0 - 3 = 0$

$\Rightarrow \vec{a} \perp \vec{b}$

Also: $\vec{b} \cdot \vec{c} = -6 + 0 + 6 = 0$

$\vec{a} + \vec{b} = \langle 4, 0, 2 \rangle$

$(\vec{a} + \vec{b}) \cdot \vec{c} = \langle 4, 0, 2 \rangle \cdot \langle -2, 0, -6 \rangle = -8 + 0 - 12$

$= \boxed{-20}$

$\boxed{\text{Ch. 11}}$