Optimization Practice

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Math 105

Problems

Question 1. Find all local extrema of the function $f(x, y) = x^2 + y^3 + xy + 5x$.

Question 2. Let $f(x, y) = x^2 + y^2 - 2x + 2$. Find the largest and smallest values $f(x, y)$ attains over the region $R = \{(x, y) : x^2 + y^2 \leq 4 \text{ and } y \geq 0\}$.

Question 3. Find and classify all local extrema of the function $f(x, y) = x^5 + y + \sin y$.

Question 4. What point on the line $y = 2x - 9$ is closest to the point $(1, 1)$?

Question 5. To manufacture a gear with outer radius R and inner radius r costs $(R-r)^2 - \ln R + 10$ dollars. A company needs a gear with $R \geq 1 + r$ (so the gear is not thin and fragile), $R \leq 5$ (so the gear fits in its housing), and $r \geq 1$ (so the gear can be mounted). What are the radii of the cheapest gear, and how much does it cost to manufacture?

Question 6. Find the maximum of the objective function $f(x, y) = 5x - 3y$ over the constrained function $g(x, y) = x^2 + y^2 - 136 = 0$.

Solutions

Solution to Question 1:

Find all local extrema of the function $f(x, y) = x^2 + y^3 + xy + 5x$.

To find local extrema, first we find all critical points, then we use the Second Derivative Test on them.

Find critical points

To find the critical points, we find the partial derivatives. Critical points are where BOTH partials are zero, or where at least one partial doesn't exist.

$$f_x = 2x + y + 5 \qquad\qquad f_y = 3y^2 + x$$

Both of these functions exist everywhere, so our only critical points occur where they are both zero. That is, when the following system is solved:

$$\begin{cases} 2x + y + 5 & = 0 \\ 3y^2 + x & = 0 \end{cases}$$

We can solve this system by substitution. If we solve the bottom equation for x, we see that for the bottom equation to be true, we must have $x = -3y^2$. For the top equation to be true at the same time, we can plug in for x:

$$2(-3y^2) + y + 5 = 0$$

Simplifying, this says $-6y^2 + y + 5 = 0$. We use the quadratic formula to find its solutions:

$$\frac{-1 \pm \sqrt{1 - 4(-6)(5)}}{2(-6)} = \frac{-1 \pm \sqrt{121}}{-12} = \frac{-1 \pm 11}{-12} = \{1, -5/6\}$$

So, our potential solutions are $y = 1$ and $y = -5/6$.

Now we can plug y into our partial derivatives to find what x should be.

If $y = 1$, then f_x tells us we should have $2x + (1) + 5 = 0$, hence $x = -3$. Checking, the point $(-3, 1)$ ALSO makes $f_y = 0$. So $(-3, 1)$ is a critical points.

If $y = -5/6$, then f_x tells us we should have $2x + (5/6) + 5 = 0$, hence $x = -25/12$. Checking, the point $(-25/12, 5/6)$ also makes $f_y = 0$. So this is the other critical point.

$\boxed{\text{critical points: } (-3, 1) \text{ and } (-25/12, -5/6).}$

Test critical points

We will test the critical points using the Second Derivative Test. We'll need to use the discriminant, $D = f_{xx}f_{yy} - (f_{xy})^2$, so we need our second derivatives.

$$f_{xx} = 2 \qquad f_{xy} = 1 \qquad f_{yy} = 6y$$

Then

$$D(x, y) = 12y - 1$$

When $y = 1$, the discriminant is $12 - 1 > 0$, and $f_{xx} = 2 > 0$, so by the Second Derivative Test:

$\boxed{(-3, 1) \text{ is a local minimum}}$

When $y = -5/6$, the discriminant is $12(-5/6) - 1 < 0$. So, by the Second Derivative Test, $(-25/12, -5/6)$ is a saddle point.

Solution to Question 2

Let $f(x,y) = x^2 + y^2 - 2x + 2$. Find the largest and smallest values $f(x,y)$ attains over the region $R = \{(x,y) : x^2 + y^2 \leq 4$ and $y \geq 0\}$, and give their locations.

Absolute maxima and minima will occur at critical points, and along boundaries. So, we need to find the critical points and evaluate f there, then evaluate f along the boundaries.

Critical Points

To find the critical points, we take the partial derivatives.

$$f_x = 2x - 2 \qquad f_y = 2y$$

The only critical point is $(1,0)$. Note $(1,0)$ IS inside the region R. $\boxed{f(1,0) = 1.}$

Boundary, one part: $x^2 + y^2 = 4$

Now it's time to test the boundary. Note the boundary is a half circle. It is given in two part: the top part that looks like half a circle, and the bottom part that is a flat line.

Suppose we only plug in values from the set $\{(x,y) : x^2 + y^2 = 4\}$. Then $f(x,y) = (x^2 + y^2) - 2x + 2 = 4 - 2x + 2 = 6 - 2x$. This is a simple function to optimize: it is largest when x is smallest (most negative), and vice-versa. Since x lies on the half circle of radius 2, $-2 \leq x \leq 2$. So $f(x,y)$ is between $6 - 2(2) = 4$ and $6 - 2(-2) = 10$.

Now, the problem asks for the extrema AND their location. So it's not enough to know x, we have to know y as well. Since we're on the line $x^2 + y^2 = 4$, we calculate that when $x = \pm 2$, $y = 0$. So, along this boundary, we have smallest value $\boxed{f(2,0) = 2}$ and largest value $\boxed{f(-2,0) = 10}$.

Boundary, other part: $y = 0$

Now let's consider what happens to the function along the lower boundary of our region, $y = 0$. Note that since $x^2 + y^2 \leq 4$, when $y = 0$, x ranges from -2 to 2.

If $y = 0$, then $f(x,y) = f(x,0) = x^2 - 2x + 2$. We want to find the maximum and minimum values of this function over the range $-2 \leq x \leq 2$. Because the function is quadratic, we know it's a parabola. To get a better idea of what it looks like, we complete the square:

$$f(x,0) = x^2 - 2x + 2 = (x-1)^2 + 1$$

So this is a parabola pointing up, with minimum value at $x = 1$. Then its maximum over $[-2,2]$ is in the location farthest from $x = 1$: that is, at $x = -2$.

So, our extrema along the boundary $y = 0$ will be when $x = 1$ and when $x = -2$. $\boxed{f(1,0) = 1}$ and $\boxed{f(-2,0) = 10}$.

Compare

Now we've found all our candidates for max and min, so we just choose the largest and smallest.

$\boxed{\text{The absolute maximum of } f(xy) \text{ over } R \text{ is 10, at } (-2,0).}$

$\boxed{\text{The absolute minimum of } f(xy) \text{ over } R \text{ is 1, at } (1,0).}$

Solution to Question 3

Find all local extrema of the function $f(x, y) = x^5 + y + \sin y$.

To find the local extrema, we find all critical points, then categorize them.

Find Critical Points

$$f_x = 5x^4 \qquad f_y = 1 + \cos y$$

$f_x = 0$ only when $x = 0$, and $f_y = 0$ only when $\cos y = -1$, which occurs for every point $y = (1 + 2n)\pi$, where n is an integer.

So, our critical points are the infinite set $\boxed{\{(0, (1 + 2n)\pi) : n \text{ integer}\}}$

Test Critical Points

To use the Second Derivative Test, we'll need the second partial derivatives.

$$f_{xx} = 20x^3 \qquad f_{xy} = 0 \qquad f_{yy} = -\sin y$$

When we're at a critical point, $x = 0$, so $f_{xx} = f_{xy} = 0$, so $D = 0$, so the Second Derivative Test is inconclusive. Oopsey-doodle. Now we have nothing left but our wits and cunning to figure out what these critical points are.

Suppose we're at a critical point $(0, y)$. If this is a max or a min, then if we hold y steady and move x around a little bit, the function should not go up (if it's a max) or not go down (if it's a min). However, since x^5 is an odd function, if we move x to make it a little bigger than 0, the function will get bigger; and if we move x to make it a little smaller than 0, the function will get smaller. So, we are neither at a local max nor a local min.

$(0, (1+2n)\pi)$ is a saddle point for every integer n. $\boxed{\text{There are no local maxima or minima–so no local extrema.}}$

Solution to Question 4

What point on the line $y = 2x - 9$ is closest to the point $(1, 1)$?

First, we should recognize this as a candidate for the Lagrange multiplier method. What we want to optimize is distance. Our constraint is that, instead of considering the distance from $(1, 1)$ to any old (x, y) in the plane, we only want point (x, y) such that $y = 2x - 9$.

Set up objective and constraint functions

The thing we want to minimize is the distance. The distance from any point (x, y) to the point $(1, 1)$ is $\sqrt{(x-1)^2 + (y-1)^2}$. However, there is a common trick: if we minimize $(x-1)^2 + (y-1)^2$, the minimum point will ALSO be the minimum point of $\sqrt{(x-1)^2 + (y-1)^2}$. So, we make things a little easier on ourselves by minimizing the alternate function $(x-1)^2 + (y-1)^2$. $\boxed{f(x, y) = (x-1)^2 + (y-1)^2}$ is our objective function.

Now, we don't care about every pair (x, y): we only care about those along the line $y = 2x - 9$. Recall: this gives us a function of one variable in a function-of-two-variables world. We need to rearrange it to be $y - 2x + 9 = 0$. So our constraint function is $\boxed{g(x, y) = y - 2x + 9}$, where we want $g(x, y) = 0$.

Find points where $g(x, y)$ is tangent to the level curve of $f(x, y)$.

This is where we set up and solve the equation $\nabla f(x, y) = \lambda \nabla g(x, y)$. The points that make this equation true for some λ will be our candidates for extrema.

We need our first partial derivatives in order to calculate the gradients.

$$f_x = 2x - 2 \qquad f_y = 2y - 2 \qquad g_x = -2 \qquad g_y = 1$$

So our equation $\nabla f(x, y) = \lambda \nabla g(x, y)$ gives:

$$< 2x - 2, 2y - 2 >= \lambda < -2, 1 >$$

This is the same as this system:

$$\begin{cases} 2x - 2 &= \lambda(-2) \\ 2y - 2 &= \lambda(1) \end{cases}$$

Usually, the easiest way to solve this equation is to solve both equations for λ, being careful to not divide by zero, and then set the different "$\lambda =$" equations equal to each other.

The bottom equation tells us that $\lambda = 2y - 2$, and the top equation tells us $\lambda = 1 - x$. So, $2y - 2 = 1 - x$. But from our constrains, recall we only care about points on the line $y = 2x - 9$, so our solutions $2y - 2 = 1 - x$ can be rewritten as

$$2(2x - 9) - 2 = 1 - x$$

This gives us $x = 21/5$, and so (using our constraint again) $y = -3/5$.

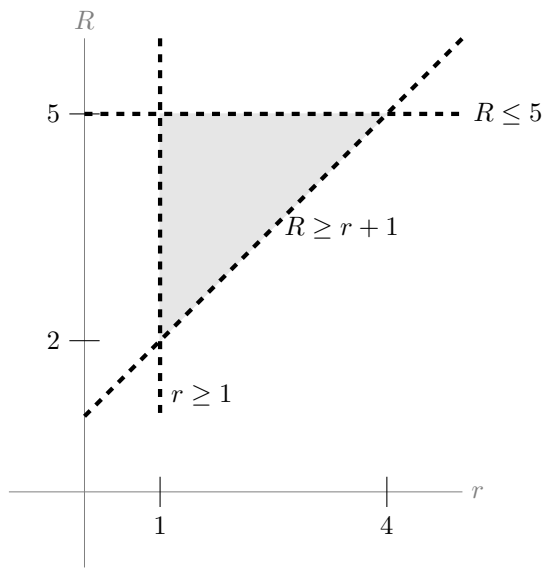
Test points

From graphing the problem, it is clear that there should be SOME point on the parabola that is closest to the point $(1, 1)$. Note, in contrast, there is no "farthest" point, since as x gets very large in the positive or negative direction, the parabola gets farther and farther away from $(1, 1)$. Therefore, our only point of interest must be the closest point on the parabola to $(1, 1)$. So $\boxed{(21/5, -3/5) \text{ is the closest point on the line } y = 2x - 9 \text{ to the point } (1, 1).}$

Solution to Question 5

To manufacture a gear with outer radius R and inner radius r costs $(R-r)^2 - \ln R + 10$ dollars. A company needs a gear with $R \geq 1 + r$ (so the gear is not thin and fragile), $R \leq 5$ (so the gear fits in its housing), and $r \geq 1$ (so the gear can be mounted). What are the radii of the cheapest gear, and how much does it cost to manufacture?

Notice that the region is a triangle.



We are trying to minimize the cost function,

$$C(r, R) = (R - r)^2 - \ln R + 10$$

The absolute minimum of this function over the appropriate region will occur at a critical point, or at a boundary point.

Critical Points

First, let's find the critical points. We need the first partial derivatives, c_r and c_R.

$$c_r = -2(R - r) \qquad c_R = 2(R - r) - \frac{1}{R}$$

When $R = 0$, the second derivative does not exist, so this would be a critical point–except it's not in the domain of the function. It's also not in our region. (This makes sense: why would we make a gear with outer radius 0?) Over our region, both partial derivatives are defined everywhere.

The partial derivative c_r is only zero when $R = r$. In this case, $c_R = 2(R - r) - \frac{1}{R} = \frac{-1}{R}$, which is never zero. So the two partial derivatives can't be zero at the same time, so there are no critical points of this type.

Therefore, there are no critical points of this function, let alone critical points in the region we are considering. The minimum must exist along the boundary.

Boundary Part 1: $r = 1$ and $2 \leq R \leq 5$

We need to test all three lines defining the region. It doesn't matter where we start, so let's jump in with the vertical line on the left. This is the line formed by the line $r = 1$, and it runs from $R = 2$ to $R = 5$.

When $r = 1$, our cost function becomes

$$c(1, R) = (R-1)^2 - \ln R + 10$$

We want the minimum of this function over the range $2 \leq R \leq 5$. But, it's not obvious! So, we need to use what we learned last semester about optimization. The minimum of this function of one variable will occur at critical points, or at endpoints.

First, let's find the critical points of the function $y = (R-1)^2 - \ln R$. We do this by differentiating: $y'(R) = 2(R-1) - \frac{1}{R}$. We set this equal to zero:

$$\begin{aligned}
y' &= 0 && \Leftrightarrow \\
2(R-1) - \frac{1}{R} &= 0 && \Rightarrow \\
R\left[2(R-1) - \frac{1}{R}\right] &= R \cdot 0 && \Rightarrow \\
2R(R-1) - 1 &= 0 && \Rightarrow \\
2R^2 - 2R - 1 &= 0 && \Rightarrow \\
R &= \frac{1 \pm \sqrt{3}}{2}
\end{aligned}$$

Since both solutions are smaller than 2, there is no critical point of the function $y = (R-1)^2 - \ln R$ in the range $[2, 5]$. Therefore, the minimum value will occur at an endpoint: $R = 2$ or $R = 5$.

$c(1, 2) = (2-1)^2 - \ln 2 + 10 = 11 - \ln 2$, and $c(1, 5) = (5-1)^2 - \ln 5 + 10 = 26 - \ln 5$. The smaller of these is $\boxed{c(1, 2) = 11 - \ln 2}$. So this is the smallest value achieved by the cost function over this particular section of the boundary.

Boundary Part 2: $R = 5$ **and** $1 \leq r \leq 4$

When $R = 5$, our cost function becomes $c(r, 5) = (5-r)^2 - \ln 5 + 10$. This equation is quadratic in r, so we notice it is a parabola, pointing up, with minimum when $r = 5$. Since $r = 5$ is not in our region, the minimum over the interval $1 \leq r \leq 4$ will occur either at $r = 1$ or $r = 4$, and it's not hard to see that the minimum occurs where $r = 4$: $\boxed{c(4, 5) = (5-4)^2 - \ln 5 + 10 = 11 - \ln 5.}$

Boundary Part 3: $R = r + 1$**,** $2 \leq R \leq 5$**, and** $1 \leq r \leq 4$

In this region, our cost function becomes

$$c(r, r+1) = (r+1-r)^2 - \ln R + 10 = 11 - \ln R$$

Since logarithm is an increasing function, $c(r, r+1)$ is larger for smaller values of R, and vice-versa. So its minimum when $2 \leq R \leq 5$ occurs when $R = 5$: $\boxed{c(4, 5) = 11 - \ln 5}$.

Compare Values

Our minima over the various boundaries is $11 - \ln 5$, which occurs for $r = 4$ and $R = 5$. So the cheapest gear has price $\boxed{11 - \ln 5, \text{ its outer radius is 5, and its inner radius is 4.}}$

Solution to Question 6

Find the maximum of the objective function $f(x, y) = 5x - 3y$ *over the constrained function* $g(x, y) = x^2 + y^2 - 136 = 0$.

Because this is a constrained optimization problem, the maximum will occur at a point when f and g are tangent: that is, when $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$.

Find Gradients

$$f_x = 5 \qquad f_y = -3 \qquad g_x = 2x \qquad g_y = 2y$$

So, $\nabla f =< 5, -3 >$ and $\nabla g =< 2x, 2y >$.

Solve $\nabla f = \lambda \nabla g$

The equation $\nabla f = \lambda \nabla y$ in this case gives us

$$< 5, -3 >= \lambda < 2x, 2y >=< \lambda \cdot 2x, \lambda \cdot 2y >$$

which is the same as the system of equations

$$\begin{cases} 5 &= \lambda \cdot 2x \\ -3 &= \lambda \cdot 2y \end{cases}$$

In general it is easiest to solve both equations for lambda. To do this, we divide by x or y; notice if $x = 0$, the top equation has no solution, and if $y = 0$, the bottom equation has no solution. So, we are free to assume $x \neq 0$ and $y \neq 0$, so we can divide to get:

$$\begin{cases} \dfrac{5}{2x} &= \lambda \\ \dfrac{-3}{2y} &= \lambda \end{cases}$$

This implies $\frac{5}{2x} = \frac{-3}{2y}$, which simplifies to $x = \frac{-5}{3}y$.

We plug this into $g(x, y)$ to solve for y:

$$\left(\frac{-5}{3}y\right)^2 + y^2 - 136 = 0$$

gives us $y = \pm 6$, and so $x = \mp 10$. That is, the points of interest are $\boxed{(10, -6) \text{ and } (-10, 6)}$.

Compare Points

The max must happen at $(10, -6)$ or $(-10, 6)$. $f(10, -6) = 68$ and $f(-10, 6) = -68$, so the maximum value is $\boxed{68}$ (and it occurs at $(10, -6)$.)