Ch 9.3 Taylor Series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots$$

Taylor Series for $f(x)$, centred at a

If $a=0$, we also call it a Maclaurin Series

ex) Find Taylor Series for $f(x) = \ln x$, centred at $x=1$.

$f(x) = \ln x$

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = -x^{-2}$

$f'''(x) = (-1)(-2)x^{-3}$

$f^{(4)}(x) = (-1)(-2)(-3)x^{-4}$

$f^{(5)}(x) = (-1)(-2)(-3)(-4)x^{-5}$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,x^{-n}$

Where does pattern start? at $n=1$

If $n=1$:

$f'(x) = (-1)^0\, 0!\, x^{-1}$

$= 1 \cdot 1 \cdot x^{-1}$ ✓

if $n=0$, $f^{(n)}(a) = f(1) = \ln(1) = \underline{\underline{0}}$

If $n \geq 1$, $\underline{\underline{f^{(n)}(a)}} = f^{(n)}(1) = \underline{\underline{(-1)^{n-1}}} (n-1)!(1)$

Taylor series: $\displaystyle\sum_{n=0}^{\infty} \frac{\boxed{f^{(n)}(a)}}{n!}(x-a)^n$

$$= 0 + \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n = \ln x$$ if it converges

(ex) Use Taylor Series we just found to approximate $\ln(9/10)$.

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$$ (if it converges!) ← depends on x

$x = 9/10$

So: $$\ln(9/10) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{9}{10}-1\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{-1}{10}\right)^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n\cdot 10^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n\cdot 10^n} = \underbrace{\sum_{n=1}^{\infty} \frac{-1}{n\cdot 10^n}}_{\text{conv or div?}}$$

$$= \underbrace{-\sum_{n=1}^{\infty} \frac{1}{n\cdot 10^n}}_{\text{consider this series (positive terms)}}$$

Notice: $\frac{1}{n\cdot 10^n} < \frac{1}{10^n}$ ← positive terms

$$\sum_{n=1}^{\infty} \frac{1}{10^n} = \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n}_{\text{geometric}} = \frac{1}{1-1/10}$$ converges

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 10^n}$$ converges by Direct Comparison Test

So $$-\sum_{n=1}^{\infty} \frac{1}{n \cdot 10^n}$$ converges too

$$\underbrace{\sum_{n=1}^{\infty} \frac{-1}{n \cdot 10^n}}_{} = \ln(9/10)$$

$$\frac{-1}{1 \cdot 10^1} + \frac{-1}{2 \cdot 10^2} + \frac{-1}{3 \cdot 10^3} + \frac{-1}{4 \cdot 10^4} + \cdots = \ln(9/10)$$

Approx: $\frac{-1}{10} = -0.1$

$$\frac{-1}{200} = \frac{-0.5}{100} = -0.005$$

$$-0.105 \approx \ln(9/10)$$

(ex)

Find Taylor Series of $f(x) = e^{2x}$, centred at $x = \frac{1}{2}\ln 2$

Definition: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

$f(x) = e^{2x} \longrightarrow f(\frac{1}{2}\ln 2) = e^{2\cdot\frac{1}{2}\ln 2} = e^{\ln 2} = 2^1$

$f'(x) = 2e^{2x} \longrightarrow f'(\frac{1}{2}\ln 2) = 2\cdot e^{2\cdot\frac{1}{2}\ln 2} = 2^2$

$f''(x) = 2^2e^{2x} \longrightarrow f''(\frac{1}{2}\ln 2) = 2^2 e^{2\cdot\frac{1}{2}\ln 2} = 2^3$

$f'''(x) = 2^3e^{2x} \longrightarrow f'''(a) = 2^4$

In general: $f^{(n)}(\frac{1}{2}\ln 2) = 2^{n+1}$ all $n \geq 0$

Taylor Series: $\boxed{\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+1}}{n!}\left(x-\frac{1}{2}\ln 2\right)^n}$

Ch 9.4 Working with Taylor Series

(compare to 9.2)

(ex) Show that $\frac{d}{dx}\{\sin x\} = \cos x$.

$$\sin x = \underline{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots} = \underline{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{-x^6}{6!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\frac{d}{dx}\{\sin x\} = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \underline{\cos x}$$

ex) Approximate e.

Note: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, all x

$\boxed{x=1}$

$e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}$

Fact: $e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

Approx: add up first several terms

$\underbrace{\frac{1}{1} + \frac{1}{1} + \frac{1}{2}}_{2.5} + \underbrace{\frac{1}{6}}_{0.1\overline{6}} + \frac{1}{24}$

$\frac{1}{24} \approx \frac{1}{25} = \frac{4}{100} = 0.04$

$$\begin{array}{r} 2.5 \\ +\ 0.1\overline{6} \\ \hline \boxed{2.\overline{6}} \end{array}$$ approx (actually $e \approx 2.718\ldots$)

ex $\displaystyle\sum_{k=1}^{\infty} \frac{(0.1)^k}{k}$

Vague Question:
What's going on w/ this series?

① Does it converge? → Yes (already said $\sum \frac{(1/10)^k}{k}$)

② To what?

From Table:

$$-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}, \quad \text{if } -1 \le x < 1$$

use $x = 0.1$

$$-\ln(1-0.1) = \underline{\underline{\sum_{k=1}^{\infty} \frac{(0.1)^k}{k}}}$$

$$\| $$

$$-\ln(9/10) = \boxed{\ln(10/9)}$$

ex $\displaystyle\sum_{k=0}^{\infty} \frac{2^k 3^{k+10}}{k!} = ?$

$$\| $$

$$\sum_{k=0}^{\infty} \frac{2^k \cdot 3^k \cdot 3^{10}}{k!} = 3^{10} \sum_{k=0}^{\infty} \frac{6^k}{k!} = \boxed{3^{10} \cdot e^6}$$

Note: $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^k}{k!}$ $\quad x=6$:

$$\sum_{k=0}^{\infty} \frac{6^k}{k!} = e^6$$

ex Evaluate $\sum_{n=0}^{\infty} n(n-1)\frac{1}{2^{n-2}}$

Start with:

$(1-x)^{-1} = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ when $|x|<1$

differentiate

$+(1-x)^{-2} = \sum_{n=0}^{\infty} n x^{n-1}$

again

$+2(1-x)^{-3} = \sum_{n=0}^{\infty} n(n-1)x^{n-2}$

use $x=\frac{1}{2}$

$2(1-\frac{1}{2})^{-3} = \sum_{n=0}^{\infty} n(n-1)\left(\frac{1}{2}\right)^{n-2}$

$=2(\frac{1}{2})^{-3}$

$=2\cdot 2^3 = \boxed{16}$

$= \left.\sum_{n=0}^{\infty} n(n-1)\cdot\frac{1}{2^{n-2}}\right\}$ STARTED WITH ex Recall:

$$2 - x^2 = ax^2 + bx + c$$

What are a, b, c:

$c = 2$

$a = -1$

$b = 0$

Can also do w/ Taylor Series

(Maclaurin)

(ex) Suppose $f(x)$ has Taylor Series

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!(2k)!}$$

What are derivatives of f at $x=0$ ← centre

$$f(x) = \underset{\frac{1}{1}}{\frac{x^0}{0!(0)!}} + \underset{\frac{1}{2}x}{\frac{x^1}{1!2!}} + \frac{x^2}{2!4!} + \frac{x^3}{3!6!} \cdots$$

$$= f(0) + f'(0)\cdot x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 \cdots$$

$(k=11)$

$$\frac{x^{11}}{11!\,22!}$$ (given)

$$\frac{f^{(11)}(0)}{11!}x^{11}$$ (def of Taylor)

$f(0)=1$
(constant term)

$f'(0) = \frac{1}{1\cdot 2} = \frac{1}{2}$

$$\frac{f'''(0)}{\cancel{3!}} = \frac{1}{\cancel{3!}6!}$$

So $f'''(0) = \frac{1}{6!}$

$$\boxed{f^{(11)}(0) = \frac{1}{22!}}$$

(ex) Taylor Series of $f(x)$ is:

$$f(x) = \sum_{k=1}^{\infty} \frac{x^{2k}}{k} = \frac{x^2}{1} + \frac{x^4}{2} + \frac{x^6}{3} + \frac{x^8}{4} + \cdots$$

$\frac{x^{12}}{6}$ $(k=6)$

Q: What is $f^{(11)}(0)$? $\boxed{=0}$

All odd derivs @ $x=0$ are 0

Q: What is $f^{(12)}(0)$?

$$\frac{1}{6} = \frac{f^{(12)}(0)}{12!}$$

So $\boxed{f^{(12)}(0) = \frac{12!}{6}}$

Taylor Series: $f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$

(Maclaurin)

$$\frac{f^{(12)}(0)}{12!}x^{12}$$