- Please put Quiz 6 in the box

   ← if you have it
   ↓

- MLC open through exam period

- I'll figure out Quiz 6 return + email you (prob: Friday)

- Office hrs on website

TS of $\ln x$:

$f^{(n)}(a)$

$$0+\sum_{n=1}^{\infty}\frac{\boxed{(-1)^{n-1}(n-1)!}}{n!}(x-1)^n$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x-1)^n \quad = \frac{1}{1}(x-1)^1+\frac{-1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4 \cdots$$

Use to approximate $\ln(9/10)$:

$$\ln x = \sum \frac{(-1)^{n-1}}{n}(x-1)^n$$

$\boxed{x=9/10}$

$$\ln\left(\frac{9}{10}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{9}{10}-1\right)^n=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{-1}{10}\right)^n$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(-1)^n\left(\frac{1}{10}\right)^n=\sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{n}\left(\frac{1}{10}\right)^n$$

odd

$$=\sum_{n=1}^{\infty}\frac{-1}{n\cdot 10^n}$$

$$\ln(9/10) \approx \underbrace{\frac{-1}{10} + \frac{-1}{200}} + \frac{-1}{3000}$$

$$-0.1 - \frac{1}{2}\left(\frac{1}{10}\right)$$

$$-[0.1 + 0.005] = -0.105$$

(ex) Find Taylor Series of $f(x) = e^{2x}$ centered at $a = \frac{1}{2}\ln 2$.

$f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

$f''(x) = 2^2 e^{2x}$

$f'''(x) = 2^3 e^{2x}$

$f^{(4)}(x) = 2^4 e^{2x}$

$\vdots$

$f(\frac{1}{2}\ln 2) = e^{2(\frac{1}{2}\ln 2)} = e^{\ln 2} = 2^1$

$f'(a) = 2 \cdot e^{2a} = 2 \cdot 2 = 2^2$

$f''(a) = 2^2 e^{2a} = 2^2 \cdot 2 = 2^3$

$f^{(n)}(\frac{1}{2}\ln 2) = 2^{n+1}$

For all $n \geq 0$

Ch 9.3 Taylor Series

$$\sum_{n=0}^{\infty} \frac{\boxed{f^{(n)}(a)}}{n!}(x-a)^n \quad \Big\} \text{ Taylor Series of } f(x) \text{ centred at } a$$

If $a=0$: "Maclaurin Series"

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots$$

ex) Find Taylor Series of $f(x) = \ln x$, centred at $a=1$.

$f(x) = \ln x$

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = (-1)x^{-2}$

$f'''(x) = (-1)(-2)x^{-3}$

$f^{(4)}(x) = (-1)(-2)(-3)x^{-4}$

$f(1) = 0$

$f'(1) = 1 = (-1)^0 \cdot 0!$

$f''(1) = (-1) = (-1)^1 \cdot 1!$

$f'''(1) = (-1)(-2) = (-1)^2 \cdot 2!$

$f^{(4)}(1) = (-1)(-2)(-3)$

$= (-1)^3 \cdot 3!$

$f^{(n)}(1) = (-1)^{n-1}(n-1)!$

if $n \geq 1$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{\boxed{2^{n+1}}}{n!}\left(x - \frac{1}{2}\ln 2\right)^n$$

(ex) Show $\frac{d}{dx}\{\sin x\} = \cos x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(-1)^n$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{-x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}(-1)^n$$

$$\frac{d}{dx}\{\sin x\} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots = \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{(2n+1)!}(-1)^n$$

$$\frac{3}{3!} = \frac{\cancel{3}}{\cancel{3}\cdot 2\cdot 1} = \frac{1}{2!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}(-1)^n$$

$$= \cos x$$

(ex) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Approximate e.

$e = e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}$

$(x=1)$

Convention:

$0! = 1$

$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$

$= \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots$

$e \approx 2 + 0.5 + 0.1\overline{6}$

$$\begin{array}{r} 2.5 \\ +\ 0.1\overline{6} \\ \hline 2.\overline{6} \end{array}$$

$e \approx 2.\overline{6}$

$e \approx 2\,{}^{2}/_{3}$

(Actually:

$e \approx 2.718\ldots$)

(ex) $\displaystyle\sum_{n=1}^{\infty} \frac{(0.1)^n}{n}$ What's going on with this series?

Could ask: Conv / Div

$\frac{(0.1)^n}{n} < (0.1)^n$

$\underbrace{\sum (0.1)^n}_{\text{CONV}}$ geometric $r = 0.1$ $|r| < 1$

Both series have positive terms

By Direct Comparison Test, $\sum \frac{(0.1)^n}{n}$ conv.

Could ask: What does it converge to?

- ~~geometric series~~
- ~~telescoping series~~
- Taylor Series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$$

$x = 0.1$

$$\sum_{n=1}^{\infty} \frac{(0.1)^n}{n} = -\ln(1-0.1)$$

$$= -\ln(9/10) = \boxed{\ln\left(\frac{10}{9}\right)}$$

ex What does $\sum_{k=0}^{\infty} k(k-1)\frac{1}{2^{k-2}}$ converge to?

Use Taylor Series

geometric $1+r+r^2+r^3+\dots$

$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ if $|r|<1$

$(1-x)^{-1} = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

differentiate both

$+(1-x)^{-2} = \sum_{k=0}^{\infty} k x^{k-1}$

diff again

$+2(1-x)^{-3} = \sum_{k=0}^{\infty} k(k-1)x^{k-2}$

$x = 1/2$

$2(1-\frac{1}{2})^{-3} = \sum_{k=0}^{\infty} k(k-1)\left(\frac{1}{2}\right)^{k-2}$

$2(1-\frac{1}{2})^{-3} = 2\left(\frac{1}{2}\right)^{-3} = 2\cdot 2^3$

$= 16$

(ex) Suppose

$$f(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{n}$$ is Taylor Series

All odd derivs are 0

- What is $f'(0)$? $\boxed{0}$ (no x)
- What is $f^{(12)}(0)$?

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{n} = \frac{x^2}{1} + 0x^3 + \frac{x^4}{2} + 0x^5 + \frac{x^6}{3} + 0x^7 + \frac{x^8}{4} + \cdots + \frac{x^{12}}{6} + \cdots$$

$$= f(0) + \boxed{f'(0)(x)} + \frac{f''(0)x^2}{2} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \cdots$$

$$+ \cdots \frac{f^{(12)}(0)}{12!} x^{12}$$

$$\frac{1}{6} = \frac{f^{(12)}(0)}{12!}$$

So $\boxed{f^{(12)}(0) = \frac{12!}{6}}$