Telescoping Series

(lots of cancellations)

(ex) $$\sum_{n=1}^{\infty}\left(\frac{1}{n+2}-\frac{1}{n+3}\right) = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(\frac{1}{3}-\frac{1}{N+3}\right) = \boxed{\frac{1}{3}}$$

$n=1$	$\frac{1}{3} - \cancel{\frac{1}{4}}$
$n=2$	$+\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}$
$n=3$	$+\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}$
$n=4$	$+\cancel{\frac{1}{6}} - \cancel{\frac{1}{7}}$
$n=5$	$+\cancel{\frac{1}{7}} - \frac{1}{8}$
	$\vdots$

$S_1 = \frac{1}{3} - \frac{1}{4}$

$S_2 = \frac{1}{3} - \frac{1}{5}$

$S_3 = \frac{1}{3} - \frac{1}{6}$

$S_5 = \frac{1}{3} - \frac{1}{8}$

$S_N = \frac{1}{3} - \frac{1}{N+3}$

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$

partial fractions

$\left(\text{Conv: } \frac{1}{n(n+1)} < \frac{1}{n^2} \right)$ $\qquad \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$

$n=1 \quad \frac{1}{1} - \cancel{\frac{1}{2}}$

$n=2 \quad + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$

$n=3 \quad + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$

$n=4 \quad + \cancel{\frac{1}{4}} - \frac{1}{5}$

$\vdots$

$S_N = 1 - \frac{1}{N+1}$

$\lim_{N\to\infty} S_N = 1 - 0 = \boxed{1}$

ex $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \left[\ln(n+1) - \ln n\right]$

DIVERGES

$n=1 \quad \ln 2 - \cancel{\ln 1}^{0}$

$n=2 \quad + \ln 3 - \ln 2$

$n=3 \quad + \ln 4 - \ln 3$

$n=4 \quad + \ln 5 - \ln 4$

$\vdots$

$S_4 = \ln 5$

$S_N = \ln(N+1)$

$\displaystyle\lim_{N\to\infty} S_N = \infty$

Table 9.5, p694 has examples of Taylor Series

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

Interval of Convergence: $(-\infty, \infty)$

Radius of Convergence: ∞

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$$

(ex) $\sum_{n=0}^{\infty} n!\,(x-2)^n$

For which values of x does it converge?

Consideration: If $x<2$, terms not all positive.

Consideration for divergence test:

$\lim_{n\to\infty} n!\,(x-2)^n$

FACT: diverges whenever $x \neq 2$

not obvious

If $x = 2.5$:

$\lim_{n\to\infty} n!\,(2.5-2)^n$

$= \lim_{n\to\infty} \underbrace{n!}_{\text{bigger}} \left(\tfrac{1}{2}\right)^n$ smaller

unclear what limit is when x is close to 2

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!\,(x-2)^{\cancel{n+1}}}{n!\,\cancel{(x-2)^n}} = \lim_{n\to\infty} (n+1)\underbrace{(x-2)}_{\text{fixed number}} = \pm\infty$$ if $x \neq 2$

$a_1 + a_2 + a_3 + a_4 + a_5 \ldots$

$\frac{a_2}{a_1}$ $\frac{a_3}{a_2}$

When n is big, I have to multiply a_n by a huge # to get a_{n+1}

i.e. $|a_n|$ is growing hugely

So: If $x \neq 2$, $\lim_{n\to\infty} a_n$ DNE

by DIVERGENCE TEST, $\sum n!(x-2)^n$ DIV when $x \neq 2$.

Ch 9.2 (cont'd) Manipulating Power Series

Thm 9.4 p 679
(paraphrase)

- addition & subtraction "work"

e.g. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$ for all x

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$ for all x

$[\sin x + \cos x] = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} \cdots$ for all x

- Multiplication by an appropriate power of x also "works" (ex) Find a power series that converges to: $\dfrac{x^3}{1-x}$

$$x^3\Big(\underbrace{\frac{1}{1-x}}_{\frac{1}{1-r}}\Big) = x^3 \underbrace{\sum_{n=0}^{\infty} x^n}_{\sum r^n} = \sum_{n=0}^{\infty} x^3 \cdot x^n = \sum_{n=0}^{\infty} x^{n+3}$$

conv when $|x|<1$

$$= \sum_{n=3}^{\infty} x^n$$

conv when $|x|<1$

$\frac{1}{x}\sin x$ vs $\frac{1}{x}\cos x$

(ex) $\underbrace{\frac{1}{x}}_{x^{-1}\sin x}\sin x$

$$\frac{1}{x}\Big(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots\Big)$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

Still a power series – o/c

$$\frac{1}{x}\Big(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots\Big)$$

$$= \underbrace{\frac{1}{x}} - \frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} \dots$$

not a power series

· Composition $f(g(x))$ is OK

if $g(x) = b x^m$, appropriate m

const

e.g. $\dfrac{1}{1-(7x^2)} = \sum_{n=0}^{\infty} (7x^2)^n = \underbrace{\sum_{n=0}^{\infty} 7^n x^{2n}}$

– still a power series

– converge to $\dfrac{1}{1-7x^2}$ (when it converges)

conv when

$|7x^2| < 1$

$7x^2 < 1$

$x^2 < \frac{1}{7}$

$|x| < \frac{1}{\sqrt{7}}$

$\boxed{-\frac{1}{\sqrt{7}} < x < \frac{1}{\sqrt{7}}}$ Interval of Convergence

Also: Integration & Differentiation of Power Series "works"

ex $\int_0^1 e^{x^2}\,dx$

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \text{(table 9.5)}$$

$$e^{(x^2)} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + \frac{x^2}{1} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

$n=0$:
$$\frac{x^0}{0!} = \frac{1}{1} = 1$$
convention

$$\int e^{x^2}\,dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!\,(2n+1)} = x + \frac{x^3}{1\cdot 3} + \frac{x^5}{2!\cdot 5} + \frac{x^7}{3!\,7}$$

$$\int_0^1 e^{x^2}\,dx = F(1) - F(0) = \sum_{n=0}^{\infty} \frac{1}{n!\,(2n+1)} - \cancel{\sum_{n=0}^{\infty} 0} = \sum_{n=0}^{\infty} \frac{1}{n!\,(2n+1)}$$

$$= \frac{1}{1} + \frac{1}{3} + \frac{1}{2(5)} + \frac{1}{6(10)} + \cdots$$

← add first several terms to get approximation

$$\sum_{n=0}^{\infty} \frac{1}{n!(2n+1)} = \lim_{N\to\infty} \underbrace{\sum_{n=0}^{N} \frac{1}{n!(2n+1)}}_{=\,?}$$

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ex) Find a power series that converges to $\arctan x$.

Notice: $\arctan x = \int \underbrace{\frac{1}{1+x^2}}_{\text{close to } \frac{1}{1-x}} dx + C$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$\frac{1}{1-r}$ $(r = -x^2)$ $\qquad \frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ when $|r| < 1$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Interval of Conv:

$|-x^2| < 1$

$x^2 < 1$

$\boxed{-1 < x < 1}$

$$\arctan x + C = \int \frac{1}{1+x^2}\,dx = \int \left(\sum_{n=0}^{\infty} (-1)^n x^{2n}\right) dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Last Step: check $+C$

Plan: plug in $x=0$

Know: $$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

If $x=0$: $$\arctan 0 = \underbrace{\left(\sum_{n=0}^{\infty} \underbrace{(-1)^n \frac{0^{2n+1}}{2n+1}}_{0} \right)}_{0} + C$$

$$\arctan 0 = C$$
$$0 = C$$

So: $$\boxed{\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$

will converge when

$-1 < x < 1$

(ex) Find a power series converging to:

$$\ln|x-1|$$

Note: $\displaystyle\int \frac{1}{x-1}\,dx = \ln|x-1| + C$

$$\int \underbrace{\frac{1}{1-x}}_{\frac{1}{1-r}}\,dx = \int \frac{-1}{x-1}\,dx$$

So: $\displaystyle \ln|x-1| = -\int \underbrace{\frac{1}{1-x}}_{\Sigma x^n}\,dx \quad + C$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

So $$\underbrace{\int \frac{1}{1-x}\,dx}_{= -\ln|x-1|+C} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$\ln|x-1| = -\sum_{n=1}^{\infty} \frac{x^n}{n} + C$$

$$= \left(\sum_{n=1}^{\infty} -\frac{x^n}{n}\right) + C$$

So: $$\boxed{\ln|x-1| = \sum_{n=1}^{\infty} -\frac{x^n}{n}}$$

To find C, set $x=0$:

$$\ln|0-1| = \left(\underbrace{\sum_{n=1}^{\infty} -\frac{0^n}{n}}_{0}\right) + C$$

$$0 = C$$

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n}$

If $x = \frac{1}{3}$:

$$\ln\left|\frac{1}{3}-1\right| = \sum_{n=1}^{\infty} -\frac{(1/3)^n}{n} = \sum_{n=1}^{\infty} -\frac{1}{n \cdot 3^n}$$

$$-\ln\left|\frac{1}{3}-1\right| = \underline{\underline{\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n}}}$$

$\parallel$

$$-\ln\left|\frac{-2}{3}\right| = -\ln\left|2/3\right| = \boxed{\ln(3/2)} = \ln 3 - \ln 2$$