1. Please be careful:
   there is a mess on
   the floor near the front
   →

2. Your quizzes are available
   @ MLC upstairs

3. Remember Q6 is due Thursday 4 PM
   (+ our last class is Wednesday)

(ex) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^{2n+1}}{3^n}$ What is the interval of convergence? (like: domain)

Looks like geometric series

Recall: $\displaystyle\sum_{n=0}^{\infty} r^n$ conv if $|r|<1$
div if $|r|\geq 1$

$$\sum_{n=0}^{\infty} \frac{(x-1)^{2n+1}}{3^n} = \sum_{n=0}^{\infty} \frac{(x-1)\cdot(x-1)^{2n}}{3^n}$$

$$= (x-1)\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{3^n} = (x-1)\sum_{n=0}^{\infty} \frac{[(x-1)^2]^n}{3^n}$$

$$= (x-1)\sum_{n=0}^{\infty} \Big(\underbrace{\frac{(x-1)^2}{3}}_{r}\Big)^n$$

$\displaystyle\sum_{n=100}^{\infty} a_n$ conv/div

EXACTLY IF (Fine print)

$\displaystyle\sum_{n=0}^{\infty} a_n$ conv/div

Conv when:

$$|r| < 1$$

$$\left| \frac{(x-1)^2}{3} \right| < 1$$

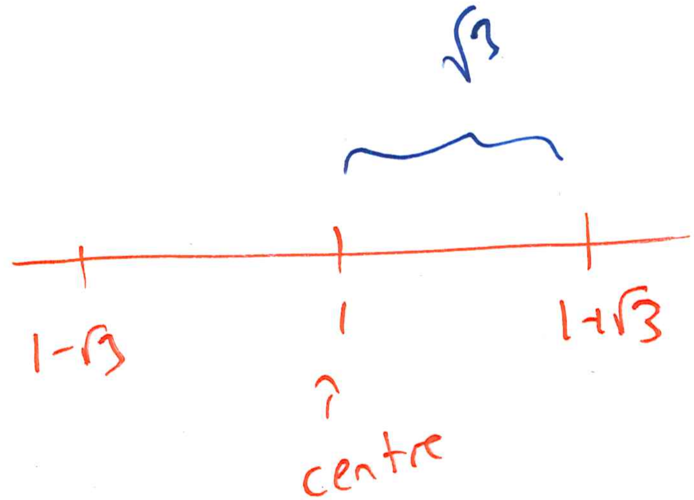
$$\frac{(x-1)^2}{3} < 1$$

$$(x-1)^2 \leq 3$$

$$|x-1| < \sqrt{3}$$

$$-\sqrt{3} < x-1 < \sqrt{3}$$

$$1-\sqrt{3} < x < 1+\sqrt{3}$$



Interval of Conv:

$$(1-\sqrt{3},\ 1+\sqrt{3})$$

Radius of Convergence:

$$\sqrt{3}$$

ex $\sum_{n=0}^{\infty} n!(x-2)^n$

What is radius of conv?

$= 1 + 1(x-2)^1 + 2!(x-2)^2 + 3!(x-2)^3$

What is interval of conv?

Divergence Test:

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=a}^{\infty} a_n$ DIVERGE

$$\lim_{n\to\infty} n!(x-2)^n = \begin{cases} \text{DIV} & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}$$

By Divergence Test, Int of Conv: $[2,2]$ (only $x=2$)

Radius of Conv: $R=0$

Another Way to Think:

Think about ratio test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(x-2)^{\cancel{n+1}}}{n!\,(x-2)^{\cancel{n}}}\right|$$

$$= \lim_{n\to\infty}(n+1)\,|x-2| = \infty$$

So: sequence $\{(n)!\,(x-2)^n\}$: growing (absolute value)

So: $\lim_{n\to\infty} n!(x-2)^n \neq 0$ if $x \neq 2$

By Divergence Test $\sum_{n=0}^{\infty} n!(x-2)^n$ DIVERGES

except when $x=2$

Manipulation of Power Series

Theorem 9.4, p 679

$\sum c_k x^k = f(x)$ on I ← interval of convergence

$\sum d_k x^k = g(x)$ "

① $\sum (c_k + d_k) x^k = f + g$ on I addition

② If $h(x) = bx^m$ (b const, m positive integer)

$\sum c_k (bx^m)^k = f(h(x))$ composition

③ If m is an integer, $k + m \geq 0$ for all terms of $\sum c_k x^k$,

then $\sum c_k x^{m+k} = x^m f$ when $x \neq 0$ multiply by a power of x

(ex) Give a power series that converges to $\sin x + \cos x$.

$$\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\right)}_{\sin x} + \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\right)}_{\cos x}$$

$$= 1 + x - \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} = \sin x + \cos x$$

for all x

(ex) $$\frac{1}{1-7x^3} = \frac{1}{1-(7x^3)} = \sum_{n=0}^{\infty} (7x^3)^n = \sum_{n=0}^{\infty} 7^n x^{3n}$$

when $|7x^3| < 1$

$|x^3| < 1/7$

$|x| < \frac{1}{\sqrt[3]{7}}$

Int of Conv:

$\left(\frac{-1}{\sqrt[3]{7}}, \frac{1}{\sqrt[3]{7}}\right)$

(ex) $$\frac{1}{1-\sin x} = \underbrace{\sum (\sin x)^n}_{\text{not a power series}}$$

(still an ok function)

(ex) $$\frac{x^3}{1-x} = x^3\left(\frac{1}{1-x}\right) = x^3 \underbrace{\sum_{n=0}^{\infty} x^n}_{\text{if } |x|<1} = \sum_{n=0}^{\infty} x^{n+3}$$

if $|x|<1$

Integrating & Differentiating also work the way we would want.

(ex) I want a power series that converges to arctan x.

Note: $\arctan x = \int \frac{1}{1+x^2}\,dx + C$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n \quad \text{if } |-x^2|<1 \text{ i.e. } |x|<1$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$\Downarrow$

$$\int \frac{1}{1+x^2}\,dx = \int \left(\sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx$$

$$= \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \right) + C$$

$\Downarrow$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

Find C: $x = $ centre (c)

To find C, plug $x=0$

$$\arctan 0 = \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{0^{2n+1}}{2n+1}}_{0} + C$$

$$0 = C$$

So: $$\boxed{\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$

Ex) Evaluate $\int_0^1 e^{x^2} dx$

Problem: $\int e^{x^2} dx$ not an elementary function

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{(list)}$$

So $$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$\int e^{x^2} dx = \int \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(2n+1)} + C$$

$$\int_0^1 e^{x^2}\,dx = \left(\sum_{n=0}^{\infty} \frac{1}{n!(2n+1)}\right) - \left(\sum_{n=0}^{\infty} \frac{0^{2n+1}}{n!(2n+1)}\right)\Bigg|_0^1$$

$$F(1) \quad - \quad F(0)$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{1}{n!(2n+1)}}$$

ex $\ln|x-1| = \sum_{k=1}^{\infty} \frac{x^k}{k}$ (show by integrating $\frac{1}{1-x}$)

Evaluate :

$\sum_{k=1}^{\infty} \frac{1}{k \cdot 3^k}$

$\boxed{\text{Let } x = \frac{1}{3}}$

$\ln\left|\frac{1}{3}-1\right| = \sum_{k=1}^{\infty} \frac{(1/3)^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k \cdot 3^k}$

So our series converges to

$\ln\left|\frac{1}{3}-1\right| = \ln\left|-\frac{2}{3}\right| = \boxed{\ln(2/3)} = \ln 2 - \ln 3$