Please Remember to fill out Course Evaluations
(comment on assessments)

from last time:

Ratio Test:

Let $\sum a_n$ be an infinite series, with positive terms, and let

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Idea: comparing mystery series to geometric series

① If $0 \le r < 1$, $\sum a_n$ converges

② If $r > 1$ (including $r=\infty$), $\sum a_n$ diverges
(think about divergence test)

③ If $r=1$, use another test

Works nicely w/ factorials; mixture of exponential & polynomial, e.g. $\sum \frac{k^2}{4^k}$

ex $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{1}{n!}}_{a_n} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$

Conv or Div ?

Try Ratio Test :

$$r = \lim_{n\to\infty} a_{n+1}/a_n = \lim_{n\to\infty} \frac{1}{(n+1)!} \Big/ 1/n!$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{\cancel{n!}}{(n+1)\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

By Ratio Test (note: positive terms)

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ <u>converges</u>.

$(n+1)! =$

$(n+1)\underbrace{n(n-1)(n-2)\cdots(1)}_{n!}$

e.g. $5! =$

$5\cdot\underbrace{4\cdot 3\cdot 2\cdot 1}_{} =$

$5\cdot 4!$

$(n+2)! =$

$(n+2)(n+1)n(n-1)\cdots(1)$

$= (n+2)(n+1)\,n!$

(ex) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{n!\,n!}{(2n)!}}_{a_n}$

Try Ratio Test:

- terms are positive
- $\displaystyle r = \lim_{n\to\infty} a_{n+1}/a_n = \lim_{n\to\infty} \frac{(n+1)!\,(n+1)!}{[2(n+1)]!} \Big/ \frac{n!\,n!}{(2n)!}$

$$= \lim_{n\to\infty} \frac{(2n)!}{n!\,n!} \cdot \frac{(n+1)!\,(n+1)!}{[2n+2]!}$$

$$= \lim_{n\to\infty} \frac{\cancel{(2n)!}}{\cancel{n!}\,\cancel{n!}} \cdot \frac{(n+1)\cancel{n!}\,(n+1)\cancel{n!}}{(2n+2)(2n+1)\cancel{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

By Ratio Test, $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,n!}{(2n)!}$ converges.

ex $\displaystyle\underbrace{\sum_{n=1}^{\infty} \frac{n^5}{(-2)^n}}_{\text{has negative terms: can't directly apply ratio test}}$

Slightly modify:

$$\sum_{n=1}^{\infty} \left|\frac{n^5}{(-2)^n}\right| = \underbrace{\sum_{n=1}^{\infty} \frac{n^5}{2^n}}_{\substack{\text{use ratio test} \\ \text{- positive terms}}}$$

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^5}{2^{n+1}} \Big/ \frac{n^5}{2^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^5}{2^{n+1}} \cdot \frac{2^n}{n^5} = \lim_{n\to\infty} \frac{(n+1)^5}{n^5} \cdot \frac{2^n}{2^{n+1}}$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n}\right)^5}_{\downarrow\ 1^5 = 1} \cdot \frac{\cancel{2^n}}{2\cdot\cancel{2^n}} = \frac{1}{2} < 1$$

By Ratio Test

$\sum \frac{n^5}{2^n}$ converges.

By Absolute Convergence Theorem:

$\sum \frac{n^5}{(-2)^n}$ converges as well.

Telescoping Series

ex $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n+2}-\frac{1}{n+3}\right) = \lim_{N\to\infty}\left(\frac{1}{3}-\frac{1}{N+3}\right)$

$= \frac{1}{3}$

$n=1 \quad \frac{1}{3} - \cancel{\frac{1}{4}}$

$n=2 \quad + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}$

$n=3 \quad + \cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}$

$n=4 \quad + \cancel{\frac{1}{6}} - \frac{1}{7}$

$S_1 = \frac{1}{3} - \frac{1}{4}$

$S_2 = \frac{1}{3} - \frac{1}{5}$

$S_3 = \frac{1}{3} - \frac{1}{6}$

$S_4 = \frac{1}{3} - \frac{1}{7}$

$S_N = \frac{1}{3} - \frac{1}{N+3}$

ex $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1$

partial fractions

(skipping)

$n=1$	$\frac{1}{1} - \cancel{\frac{1}{2}}$
$n=2$	$+ \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$
$n=3$	$+ \cancel{\frac{1}{3}} - \frac{1}{4}$
$n=4$	$+ \frac{1}{4} - \frac{1}{5}$

$S_1 = 1 - 1/2$

$S_2 = 1 - 1/3$

$S_3 = 1 - 1/4$

$S_4 = 1 - 1/5$

$\vdots$

$S_N = 1 - 1/N+1$

$\lim_{N \to \infty} S_N = 1$

ex $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} [\ln(n+1) - \ln n]$ DIVERGES

$\lim_{n\to\infty} S_n$ not a #

$n=1$	$\cancel{\ln 2} - \ln 1$
$n=2$	$+ \cancel{\ln 3} - \cancel{\ln 2}$
$n=3$	$+ \cancel{\ln 4} - \cancel{\ln 3}$
$n=4$	$+ \cancel{\ln 5} - \cancel{\ln 4}$
$n=5$	$+ \ln 6 - \cancel{\ln 5}$
	$\vdots$

$\ln 1 = 0$

$S_1 = \ln 2$

$S_2 = \ln 3$

$S_3 = \ln 4$

$S_4 = \ln 5$

$\vdots$

$S_N = \ln(N+1)$ $\lim_{N\to\infty} S_N = \infty$

$a_n = \ln\left(\frac{n+1}{n}\right)$ $\lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = \ln 1 = 0$ Div Thm: no help

Power Series:

$$\sum_{n=0}^{\infty} c_n (x-a)^n \quad :$$

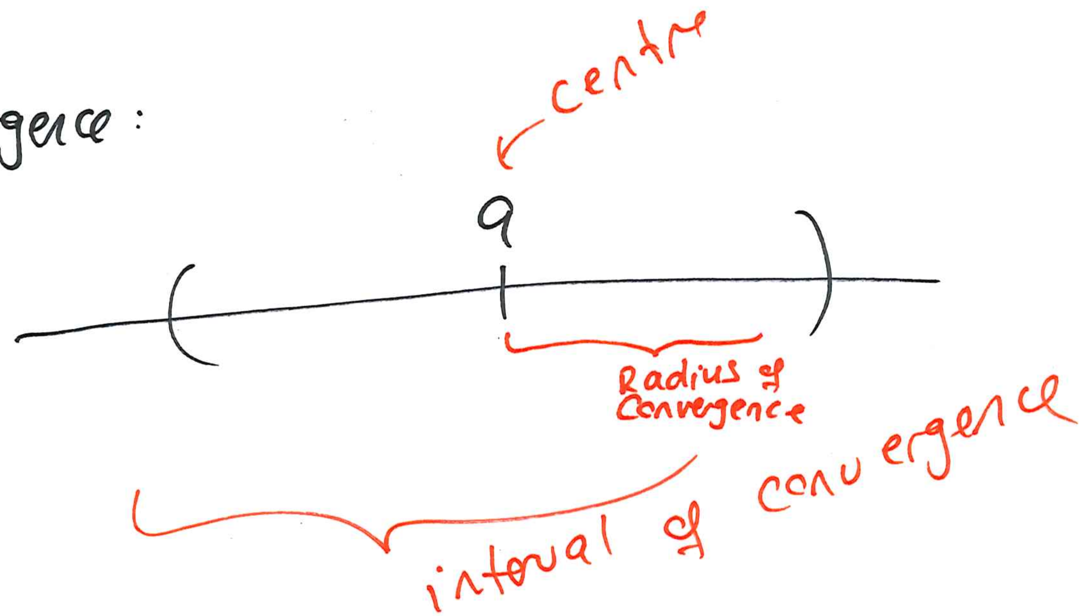
a: constant
"centre"

x: variable

$\{c_n\}$: sequence of constants (no x)

<u>Interval of convergence</u> values of x that make series <u>converge</u>

Radius of convergence:



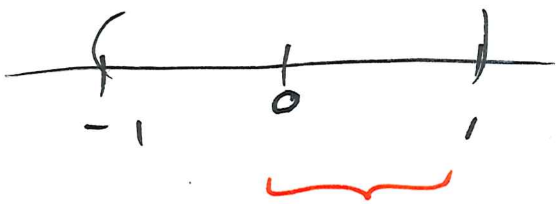
ex $\displaystyle\sum_{n=0}^{\infty} x^n$

$\{c_n\} = \{1\}$

$a = 0$

(geometric)

Interval of Convergence: $(-1, 1)$



Radius of Convergence:

$R = 1$

$\displaystyle\sum_{n=0}^{\infty} r^n$ converges for $|r| < 1$

$r = x$

$\sum x^n$ conv. for $|x| < 1$

i.e. $-1 < x < 1$

ex $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (looks like e^x)

If $x>0$, pos. terms:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}$$

$$= \lim_{n\to\infty} \frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{x}{n+1} = 0 \quad \text{for } \underline{\text{any}} \ x$$

interval of convergence: $(-\infty, \infty)$

$R = \infty$