→ Ch 9.1 → optional

Taylor Polynomials; Taylor Remainder Theorem

→ Quiz 6: Take-home

(please staple; name & SID in upper-right corner)

→ Please fill out course evaluations online

esp comment on quizzes

So far, we've learned these tests:

- Divergence Test
- Integral Test
- Direct Comparison Test
- Limit Comparison Test
- p-test

only work on series that have positive terms

Absolute Convergence Theorem!

If $\sum_{n=a}^{\infty} |a_n|$ converges, then $\sum_{n=a}^{\infty} a_n$ converges, too.

e.g. $\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}}$ Conv or Div?

$-\frac{1}{1} + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} \cdots$

terms not all positive

Consider a different series:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$

converges by p-test ($p=2$)

So, by Absolute Convergence Theorem, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges. (absolutely)

If $\sum a_n$ converges, there are two possibilities:

1. $\sum |a_n|$ converges — We say $\sum a_n$ converges absolutely

2. $\sum |a_n|$ diverges — We say $\sum a_n$ converges conditionally

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^3+1}$ Conv or Div?

Some negative terms

Consider:

$$\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^3+1}\right|$$

Note:

- $\left|\dfrac{\cos n}{n^3+1}\right| \le \dfrac{1}{n^3}$
- Both sequences have positive terms
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges ($p=3$ p-test)

So, by Direct Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^3+1}\right|$ converges.

By Absolute Convergence Theorem, $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^3+1}$ converges. (absolutely)

Ratio Test

Idea: compare mystery series to a geometric series

Let $\sum_{n=a}^{\infty} a_n$ be a series with positive terms,

and let $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

① If $0 \le r < 1$, the series converges

② If $r > 1$ (including $r = \infty$), the series diverges

③ If $r = 1$: need another test

ex $\sum_{n=1}^{\infty} \underbrace{\frac{n^2}{4^n}}_{\text{positive}}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{(n+1)^2}{4^{n+1}}\right) \Big/ \left(\frac{n^2}{4^n}\right)$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{4^{n+1}} \cdot \frac{4^n}{n^2} = \lim_{n\to\infty} \Big(\underbrace{\frac{n+1}{n}}_{1}\Big)^2 \cdot \frac{\cancel{4^n}}{\cancel{4^n} \cdot 4} = \frac{1}{4} < 1$$

So $\sum_{n=1}^{\infty} \frac{n^2}{4^n}$ converges by ratio test.

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$ Use Ratio Test

- Positive Terms
- $r = \displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(n+1)^5} \Big/ 1/n^5$

$$= \lim_{n\to\infty} \underbrace{\frac{n^5}{(n+1)^5}}_{\underbrace{\left(\frac{n}{n+1}\right)^5}_{1^5 = 1}} = \frac{1}{1} = 1$$

Ratio Test: inconclusive ($r=1$)

Use p-test: $p = 5 > 1$, so $\sum \frac{1}{n^5}$ converges.

Quick Review: factorials

$$n! = n(n-1)(n-2)\cdots(1)$$

eg.

$$4! = 4\cdot 3\cdot 2\cdot 1 = 24$$

$$5! = 5\cdot\underbrace{4\cdot 3\cdot 2\cdot 1} = 5\cdot 4! = 5\cdot 24 = 120$$

$$\frac{5!}{4!} = \frac{5\cdot\cancel{4\cdot 3\cdot 2}\cdot 1}{\cancel{4\cdot 3\cdot 2\cdot 1}} = 5$$

Similarly:

$$\frac{(n+1)!}{n!} = \frac{(n+1)\,\cancel{n(n-1)(n-2)\cdots(1)}}{\cancel{n(n-1)(n-2)\cdots(1)}} = n+1$$

$$\sum_{n=1}^{\infty} \underbrace{\frac{1}{n!}}_{a_n}$$

Conv or Div?

Ratio Test:

- positive terms
- $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{1}{(n+1)!}\right) \Big/ \frac{1}{n!}$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{\cancel{n!}}{(n+1)\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

So, by Ratio Test, $\sum_{n=1}^{\infty} \frac{1}{n!}$ <u>converges</u>

ex $\sum_{n=1}^{\infty} \frac{n^5}{(-2)^n}$ Conv or Div?

$2^{n-1} = 2^n \cdot 2^{-1} = \frac{2^n}{2}$

Can only use Ratio Test if positive terms

$$\sum_{n=1}^{\infty} \left|\frac{n^5}{(-2)^n}\right| = \sum_{n=1}^{\infty} \frac{n^5}{2^n}$$ } - positive terms - use ratio test

$2^{n+1} = 2^1 2^n = 2 \cdot 2^n$

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^5}{2^{n+1}} \Big/ \frac{n^5}{2^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^5}{2^{n+1}} \cdot \frac{2^n}{n^5}$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n}\right)^5}_{1^5=1} \cdot \frac{\cancel{2^n}}{\cancel{2^n} \cdot 2} = \frac{1}{2} < 1$$

$\frac{(n+1)^5}{n^5} = \left(\frac{n+1}{n}\right)^5$

$\frac{x^2}{y^2} = \frac{x \cdot x}{y \cdot y} = \frac{x}{y} \cdot \frac{x}{y} = \left(\frac{x}{y}\right)^2$

So $\sum \left|\frac{n^5}{(-2)^n}\right|$ converges by Ratio Test

By Absolute Convergence Theorem:

$\sum_{n=1}^{\infty} \frac{n^5}{(-2)^n}$ converges (absolutely)

Quick Review: Taylor Polynomials

Centre (a).

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\boxed{T_3(a) = f(a)} + 0 + 0 + 0$$

$$T_3'(x) = f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2}(x-a)^2$$

$$\boxed{T_3'(a) = f'(a)} + 0 + 0$$

$$T_3''(x) = f''(a) + f'''(a)(x-a)$$

$$\boxed{T_3''(a) = f''(a)} + 0$$

$$T_3'''(x) = f'''(a)$$

$$\boxed{T_3'''(a) = f'''(a)}$$

$$T_3^{(4)}(x) = 0$$

$$T_3^{(5)}(x) = 0$$

$$\vdots$$

Taylor Poly:

$$\sum_{n=1}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Series:

$$\sum_{n=1}^{\infty} \underbrace{\frac{f^{(n)}(a)}{n!}}_{\substack{\text{constants:} \\ \text{they change with } n \\ \text{no } x}}(x-a)^n$$

variable

constant

Power Series:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

$\{c_n\}$: sequence of constants (no x)

x : variable

a : constant "centre"

Vocab:

The set of values of x for which it converges: Interval of Convergence, I

The radius of convergence, R, is the distance from the centre to the boundary of the interval of convergence.

ex $\sum_{n=0}^{\infty} x^n$ Power Series (Geometric)

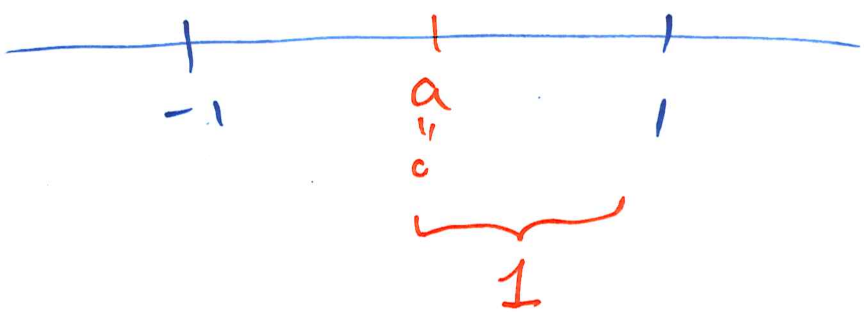
Converges when $|x|<1$ i.e. $-1<x<1$

$\boxed{(-1, 1)}$ "Interval of Convergence"

Centre: $a=0$

Radius of Convergence:

$R=1$



p694 : big list

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$R = \infty$

i.e.

Int. of Convergence

$(-\infty, \infty)$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$R = \infty$

$a = 0$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$R = \infty$

$a = 0$

ex $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ For which values of x does it converge?

$$\sum \left|\frac{x^k}{k!}\right| : \quad \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \left|\frac{x^{k+1}}{(k+1)!}\right| \Big/ \left|\frac{x^k}{k!}\right|$$

$$= \lim_{k\to\infty} \frac{|x^{k+1}|}{(k+1)!} \cdot \frac{k!}{|x^k|}$$

$$= \lim_{k\to\infty} \frac{|x|^{k+1}}{|x|^k} \cdot \frac{k!}{(k+1)!}$$

$$= \lim_{k\to\infty} \frac{|x| \cdot \cancel{|x|^k}}{\cancel{|x|^k}} \cdot \frac{\cancel{k!}}{(k+1)\cancel{k!}}$$

$$= \lim_{k\to\infty} \frac{|x|}{(k+1)} = 0$$ no matter what constant we plug in for x

So by Ratio Test:

$\sum \frac{x^k}{k!}$ converges for all x.

ex $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k+1}}{3^k}$

For which values of x does this converge?

Try to write as $\sum r^k$ (geometric)

$$\sum_{k=0}^{\infty} \frac{x \cdot x^{2k}}{3^k} = x \sum_{k=0}^{\infty} \frac{x^{2k}}{3^k} = x \sum_{k=0}^{\infty} \frac{(x^2)^k}{3^k}$$

$$= x \sum_{k=0}^{\infty} \left(\frac{x^2}{3}\right)^k$$

Geom series, $r = \frac{x^2}{3}$

Conv. when $|r| < 1$

ie $-1 < r < 1$

$$-1 < \frac{x^2}{3} < 1$$

$$-3 < x^2 < 3$$

$$|x| < \sqrt{3}$$

So: $-\sqrt{3} < x < \sqrt{3}$

Interval of Convergence:

$(-\sqrt{3}, \sqrt{3})$

Radius of Convergence:

$\sqrt{3}$

ex $\sum_{k=0}^{\infty} k!\,(x-2)^k$

Which values of x make it converge?

Factorials → Ratio

$r = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \left| \frac{(k+1)!\,(x-2)^{k+1}}{k!\,(x-2)^k} \right| = \lim_{k\to\infty} (k+1)\,|x-2|$

↑ cancellations

If $x=2$: $r=0$, so $\sum k!\,(x-2)^k$ converges

If $x\neq 2$: $r=\infty$ so $\sum k!\,(x-2)^k$ diverges

Interval of Convergence: $[2,2]$ $(x=2)$

Radius of Convergence: $R=0$

$a=2$

centre