Announcements:

- Please fill out course evaluations
  (link in email)

- Lowest webwork dropped

- Quiz 6 is take-home; due April 6, 4pm
  (on website)

TA request:
- name & SID in upper-left corner
- staple

- Ch 9.1 changed to "optional" –
  Review Taylor Polynomials on your own

Last Time:

Direct Comparison Test

Suppose $\sum_{n=a}^{\infty} a_n$ and $\sum_{n=a}^{\infty} b_n$ are series with positive terms, and $a_n \le b_n$ for all large n.

- If $\sum_{n=a}^{\infty} a_n$ diverges, then $\sum_{n=a}^{\infty} b_n$ diverges as well.
- If $\sum_{n=a}^{\infty} b_n$ converges, then $\sum_{n=a}^{\infty} a_n$ converges as well.

What if the inequalities go the wrong way?

Limit Comparison Test

Let a_n, b_n be sequences with positive terms, and

$\lim_{n\to\infty} (a_n / b_n)$ is a real number, not 0

Then: $\sum_{n=a}^{\infty} a_n$ and $\sum_{n=a}^{\infty} b_n$ either ← a: any integer where we start adding

- both converge or
- both diverge.

- A little harder than Direct Comparison Test
- Works more often.

(ex) $\displaystyle\sum_{n=1}^{\infty} \frac{1+n}{n^3+1}$ Converge or Diverge?

Idea: compare it to a nicer series
easier

$\frac{1+n}{n^3+1} \approx \frac{n}{n^3} = \frac{1}{n^2}$ Guess: $\frac{1}{n^2}$ might be a good comparison

$\left.\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}\right\}$ $p = 2 > 1$ so converges by p-test

For Direct Comparison Test: would need

$\frac{1+n}{n^3+1} \le \frac{1}{n^2}$ $(n^2 \cdot (n^3+1))$ cross-multiply

Inequality goes the wrong way

$\Updownarrow$ $n^2 + \cancel{n^3} \le \cancel{n^3} + 1$

$\Updownarrow$ $n^2 \le 1$ FALSE $n \to \infty$

Can't use Direct Comparison Test

$$\lim_{n\to\infty} \left(\frac{1+n}{n^3+1}\right) \Big/ \left(\frac{1}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{n^2+n^3}{n^3+1} = \frac{1}{1} = 1 \neq 0$$

Both series have positive terms

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-test)

So, by Limit Comparison Test,

$\sum_{n=1}^{\infty} \frac{1+n}{n^3+1}$ $\boxed{\text{converges}}$ as well.

(ex) $\displaystyle\sum_{n=10}^{\infty} \frac{1}{2\sqrt{n}-9} = \left(\frac{1}{2\sqrt{10}-9}\right) + \left(\frac{1}{2\sqrt{11}-9}\right) + \left(\frac{1}{2\sqrt{12}-9}\right) + \cdots$

conv or div?

Divergence Test:

$\displaystyle\lim_{n\to\infty} \frac{1}{2\sqrt{n}-9} = 0$ (no help)

Integral test:

$\displaystyle\int_{10}^{\infty} \frac{1}{2\sqrt{x}-9}\,dx$ looks hard

Compare:

$\displaystyle\sum_{n=10}^{\infty} \frac{1}{2\sqrt{n}} = \frac{1}{2}\sum_{n=10}^{\infty} \frac{1}{\sqrt{n}}$

$p = 1/2 < 1$

So by p-test, diverges

For Direct Comparison Test would need $\frac{1}{2\sqrt{n}-9} \geq \frac{1}{2\sqrt{n}}$

For n large: $2\sqrt{n}-9 < 2\sqrt{n}$, both positive

so $\frac{1}{2\sqrt{n}-9} > \frac{1}{2\sqrt{n}}$

- $\displaystyle\sum_{n=10}^{\infty} \frac{1}{2\sqrt{n}}$ & $\displaystyle\sum_{n=10}^{\infty} \frac{1}{2\sqrt{n}-9}$

both have positive terms

- $\displaystyle\sum_{n=10}^{\infty} \frac{1}{2\sqrt{n}}$ DIVERGES (p-test, $p = 1/2$)

- $\displaystyle\frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{n}-9}$ when n large

So, by Direct Comparison Test,

$\displaystyle\sum_{n=10}^{\infty} \frac{1}{2\sqrt{n}-9}$ also $\boxed{\text{diverges}}$

(ex) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+5}{n^4+3n+6}$ CONV OR DIV ?

Idea: compare to $\dfrac{n^2}{n^4} = \dfrac{1}{n^2}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p=2$)

I don't want to go to the trouble of deciding whether or not

$$\frac{n^2+n+5}{n^4+3n+6} \le \frac{1}{n^2}$$

I'll just use limit comparison.

- Both series have positive terms
- $\displaystyle\lim_{n\to\infty} \left(\frac{n^2+n+5}{n^4+3n+6}\right) \Big/ \left(1/n^2\right) = \lim_{n\to\infty} \frac{n^4+n^3+5n^2}{n^4+3n+6} = \frac{1}{1} = 1 \neq 0$

So, by Limit Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+5}{n^4+3n+6}$ $\boxed{\text{converges}}$ too

Absolute Convergence Theorem

If $\sum_{n=a}^{\infty} |a_n|$ converges, then $\sum_{n=a}^{\infty} a_n$ converges as well.

Def: If $\sum |a_n|$ converges, we say $\sum a_n$ is absolutely convergent.

If $\sum |a_n|$ diverges, and $\sum a_n$ converges, we say $\sum a_n$ is conditionally convergent.

(ex) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ Conv or Div?

Some terms are <u>negative</u>

$= \frac{-1}{1} + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} \cdots$

Can't use: $\int$ test
DCT
LCT

What about:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \cdots$

$p = 2 > 1$
So: <u>converges</u> (p-test)

So, by Absolute Convergence Theorem,

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges also.

Ratio Test

Idea: kind of comparing our mystery series to a geometric series

Let $\sum_{n=a}^{\infty} a_n$ have positive terms, and let

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

If $0 \le r < 1$, $\sum a_n$ converges

If $r > 1$ (including $r = \infty$), $\sum a_n$ diverges

If $r = 1$: need another test

ex $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{n^2}{4^n}}_{a_n}$ Conv or Div?

- positive terms

Take $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{(n+1)^2}{4^{n+1}}\right) \Big/ \left(\frac{n^2}{4^n}\right)$

$\displaystyle = \lim_{n\to\infty} \frac{(n+1)^2}{4^{n+1}} \cdot \frac{4^n}{n^2} = \lim_{n\to\infty} \frac{\cancel{4^n}}{\cancel{4^n} \cdot 4} \cdot \underbrace{\left(\frac{n+1}{n}\right)^2}_{\to 1^2 = 1}$

$\displaystyle = \frac{1}{4}(1) = \frac{1}{4} < 1$

So, by Ratio Test $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{4^n}$ converges

FACTORIALS

Remember: $n! = n(n-1)(n-2)\cdots(1)$

e.g. $4! = 4(3)(2)(1) = 24$

$5! = 5\cdot4\cdot3\cdot2\cdot1 = 5\cdot4! = 5\cdot24 = 120$

$$\frac{5!}{4!} = \frac{5\cdot\cancel{4\cdot3\cdot2}\cdot1}{\cancel{4\cdot3\cdot2\cdot1}} = 5$$

In general: $\dfrac{(n+1)!}{n!} = \dfrac{(n+1)\cancel{n(n-1)(n-2)\cdots(1)}}{\cancel{n(n-1)(n-2)\cdots(1)}} = n+1$

In general: $\dfrac{(n+1)!}{(n-1)!} = (n+1)(n)$

* Factorials play nicely w/ ratio test.