ANNOUNCEMENTS

1. Lowest WebWork will be dropped

2. Please fill out course evaluations

Divergence Test

If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum\limits_{n=a}^{\infty} a_n$ DIVERGES

$$\lim_{n\to\infty} a_n$$

$\neq 0$: $\sum\limits_{n=a}^{\infty} a_n$ DIVERGES

$= 0$:

Maybe $\sum\limits_{n=a}^{\infty} a_n$ converges

Maybe it diverges

We need to think more

We'll learn more tests

ex $\sum_{n=1}^{\infty} \sin n = \sin(1) + \sin(2) + \sin(3) + \cdots$

converge or diverge?

$\lim_{n\to\infty} \sin n$ DNE (jumps around)

So by Divergence Test, $\sum_{n=1}^{\infty} \sin n$ DIVERGES

ex $\sum_{n=1}^{\infty} \frac{n^2+1}{n} = \left(\frac{1+1}{1}\right) + \left(\frac{4+1}{2}\right) + \left(\frac{9+1}{3}\right) + \cdots$

$\lim_{n\to\infty} \left(\frac{n^2+1}{n}\right) = \infty \neq 0$

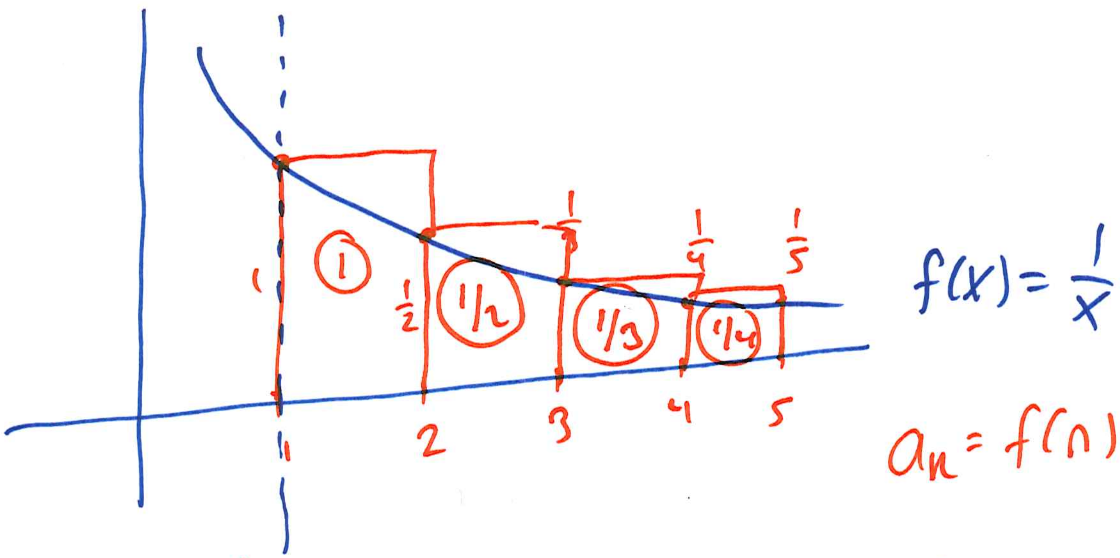
By Divergence Test, $\sum_{n=1}^{\infty} \frac{n^2+1}{n}$ DIVERGES (ex) $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

"Harmonic series"

Let's Try Divergence Test:

$\lim_{n\to\infty} \frac{1}{n} = 0$ No help!

Idea:



Area under rectangles:

$\sum_{n=1}^{\infty} \frac{1}{n} \;>$

Area under curve $f(x) = \frac{1}{x}$

$\int_1^{\infty} \frac{1}{x}\,dx = \infty$

(p-test)

So $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (DIVERGES)

Integral Test

Let $f(x)$ be a function that is:

- positive
- decreasing
- continuous

on $[a, \infty)$

and let $\{a_n\} = \{f(n)\}$

Then: $\sum_{n=a}^{\infty} a_n$ & $\int_a^{\infty} f(x)\,dx$ either:

- both CONVERGE, or
- both DIVERGE

(ex) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

can't use integral test!

$\dfrac{\cos n}{n^2}$: not all positive

not always decreasing

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ Conv or Div?

Try Divergence Test:

$$\lim_{n\to\infty} \frac{1}{n^2+1} = 0 \quad \text{(no help)}$$

Let $f(x) = \dfrac{1}{x^2+1}$

$f(x)$ is: positive, decreasing, continuous

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \left[\arctan b - \arctan 1\right]$$

$$= \pi/2 - \pi/4 = \pi/4$$

So: $\displaystyle\int_1^{\infty} \frac{1}{x^2+1}\,dx$ CONVERGES

By Integral Test, also

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ CONVERGES

Ex $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n\ln n}$ Converge or Diverge?

Let $f(x) = \dfrac{1}{x\ln x}$

$f(x)$: positive on $[10, \infty)$
decreasing
continuous on $[10, \infty)$

$$\int_{10}^{\infty} \frac{1}{x\ln x}\,dx \underset{\substack{u=\ln x \\ du=\frac{1}{x}dx}}{=} \int_{\ln 10}^{\infty} \frac{1}{u}\,du = \lim_{b\to\infty}\Big[\underbrace{\ln b}_{\to\infty} - \underbrace{\ln(\ln 10)}_{\text{some } \#}\Big]$$

$= \infty$ So integral diverges.

By the Integral Test:

$\displaystyle\sum_{n=10}^{\infty} \frac{1}{n\ln n}$ diverges as well.

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, p some positive constant

$$f(x) = \frac{1}{x^p}$$ ← positive decreasing continuous

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \text{CONVERGING} & p > 1 \\ \text{DIVERGING} & p \leq 1 \end{cases}$$

By Integral Test:

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is: convergent if $p > 1$, divergent if $p \leq 1$ } p-test for series ex $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$p = 2$
converges by p-test

ex $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

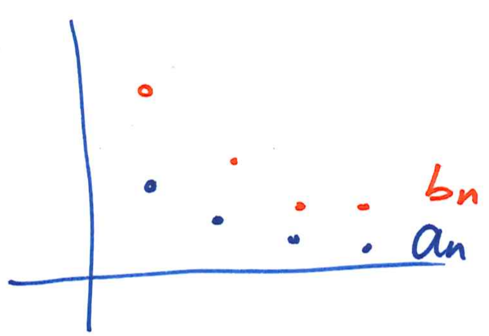
$p = 1/2$
DIVERGES by p-test

Ch 8.5 Ratio & Comparison Tests

Direct Comparison Test:

Let a_n, b_n be sequences with positive terms and $a_n \le b_n$ for all n larger than some constant.

- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.
- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges as well.



(ex) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

Does it converge or diverge?

Reasonable guess:

compare to $\displaystyle\sum \frac{1}{n^2}$ ← $p=2$ CONVERGES

Need: $\frac{1}{n^2+n} \le \frac{1}{n^2}$ (TRUE)

- $\frac{1}{n^2+n}$, $\frac{1}{n^2}$: positive
- $\frac{1}{n^2+n} \le \frac{1}{n^2}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGES by p-test

$\left.\right\}$ By Direct Comparison Test,

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ CONVERGES

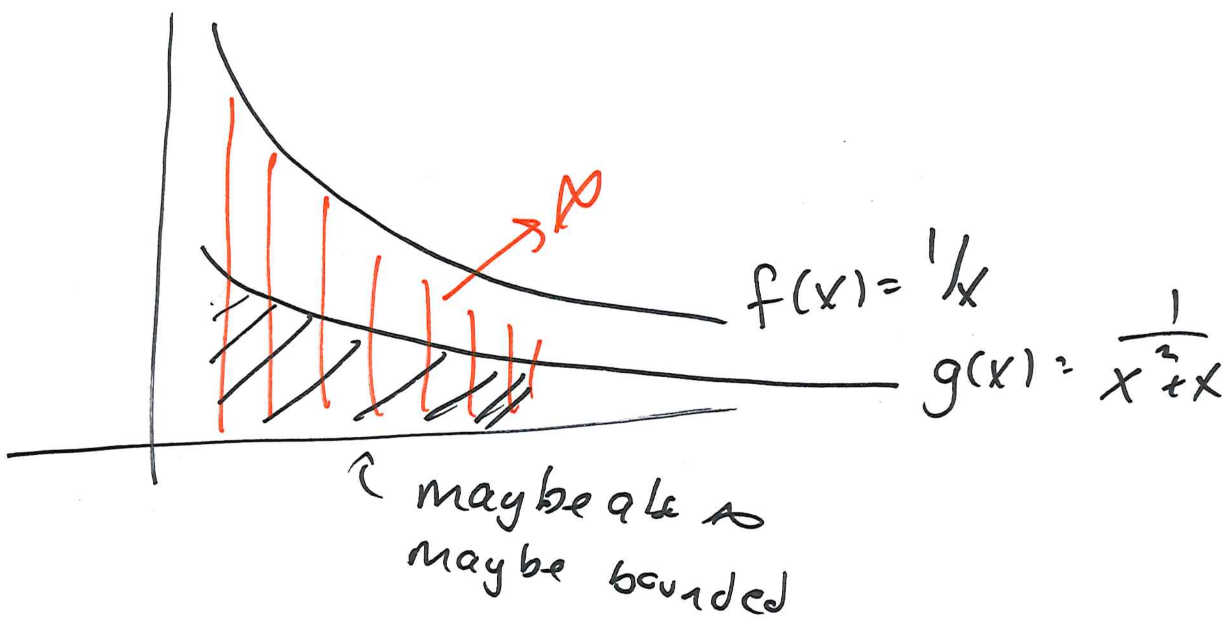
Q: What if we compare $\frac{1}{n^2+n}$ with $\frac{1}{n}$?

$$\sum_{n=1}^{\infty} \frac{1}{n} : \text{DIVERGENT} \quad \text{(Harmonic Series)}$$

$p = 1$

$$\frac{1}{n} \geq \frac{1}{n^2+n}$$

Inequality goes wrong way — can't use direct comparison test



(ex) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ Conv or div ?

Try Div Test :

$\lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = 0$ no help

Think about integral test :

$\int \frac{1}{\sqrt{x}-1}\,dx$ ← unpleasant

+
dec
cont

Notice : $\underbrace{\frac{1}{\sqrt{n}-1}}_{\text{positive}}$ looks a lot like $\underbrace{\frac{1}{\sqrt{n}}}_{\text{positive}}$

$$\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$$

$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$: $p = 1/2$, so series DIVERGES

So, by Direct Comparison Test:

$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ DIVERGES

- $\frac{1}{\sqrt{n}}$, $\frac{1}{\sqrt{n}-1}$ positive
- $\sum \frac{1}{\sqrt{n}}$ DIV
- $\frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n}-1}$

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}-9}$

Want to compare to

$$\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}} = \frac{1}{2} \underbrace{\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}}_{\substack{p=1/2 \\ \text{p-test: DIVERGES}}}$$

$n = 100$

$\frac{1}{2\sqrt{n}} = \frac{1}{20} < \frac{1}{2\sqrt{n}-9} = \frac{1}{20-9} = \frac{1}{11}$

In order to use direct comparison tests I need:

$$\frac{1}{\underline{2\sqrt{n}}} \leq \frac{1}{\underline{2\sqrt{n}-9}} \qquad \text{TRUE}$$

positive

positive for $\boxed{\text{large } n}$

By Direct Comparison Test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}-9}$ also diverges

Limit Comparison Test:

Let a_n, b_n be sequences with positive terms, and

$\lim_{n \to \infty} a_n / b_n$ is a real number (not $\pm\infty$) not 0

Then: $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ either

- both converge, or
- both diverge

What if inequality goes wrong way?

(ex) $\displaystyle\sum_{n=1}^{\infty} \frac{1+n}{n^3+1}$ conv or div?

Divergence Test: $\displaystyle\lim_{n\to\infty} \frac{1+n}{n^3+1} = 0$ no help

Integral test: partial fractions? ugh

Direct comparison Test:

$$\frac{1+n}{n^3+1} \approx \frac{n}{n^3} = \frac{1}{n^2}$$ compare to this

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ <u>converges</u> ($p=2$, p-test)

<u>NEED</u> (to use direct comparison test)

$$\frac{1+n}{n^3+1} \leq \frac{1}{n^2} \rightarrow n^2+n^3 \leq n^3+1 \quad \text{FALSE}$$

Inequality goes the wrong way!

Compare:

$$\sum \frac{1+n}{n^3+1} \quad \text{and} \quad \sum \frac{1}{n^2}$$

- $\frac{1+n}{n^3+1}$ & $\frac{1}{n^2}$: positive

- $\lim_{n\to\infty} \left(\frac{1+n}{n^3+1}\right) \Big/ \left(\frac{1}{n^2}\right) = \lim_{n\to\infty} \frac{n^2(1+n)}{n^3+1}$

$$= \lim_{n\to\infty} \frac{n^3+n}{n^3+1} = \frac{1}{1} = 1 \neq 0$$

- $\sum \frac{1}{n^2}$ converges ($p=2$, p-test)

By Limit Comparison Test, $\sum \frac{1+n}{n^3+1}$ also $\boxed{\text{converges}}$

ex $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+5}{n^4+3n+6}$ conv or div ?

Compare : $\dfrac{n^2}{n^4} = \dfrac{1}{n^2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONV, $p=2$, p-test

For Direct Comparison Test:

need $\dfrac{n^2+n+5}{n^4+3n+6} \leq \dfrac{1}{n^2}$ ← I'm lazy & I don't want to check this!

Limit Comparison Test :

$$\lim_{n\to\infty} \left(\frac{n^2+n+5}{n^4+3n+6}\right) \Big/ \left(\frac{1}{n^2}\right) = \lim_{n\to\infty} \frac{n^4+n^3+5n^2}{n^4+3n+6} = \frac{1}{1} = 1 \neq 0$$

- Both series have positive terms

- $\lim_{n \to \infty} (\quad) / (1/n^2) = 1 \neq 0$

- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test

So: by Limit Comparison Test,

$\sum_{n=1}^{\infty} \frac{n^2+n+5}{n^4+3n+6}$ also converges ex) $\sum 3(1.001)^k$

Geometric, $r = 1.001 > 1$
So DIVERGES
(can also use div test)

ex) $\sum \underbrace{\frac{n^2+1}{n+2}}_{\text{terms: } \to \infty}$

DIVERGES
(by Divergence Test)

ex) $\sum \frac{1}{e^n} = \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$

$= \frac{1}{1-1/e}$

Geometric,
$r = \frac{1}{e} < 1$
So: converges