Ch 8.4 : Divergence & Integral Tests

Recall: $S_N = \sum_{n=1}^{N} a_n$ and $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} S_N$

partial sum

terms of sequence

Idea behind divergence test:

Suppose $\sum_{n=1}^{\infty} a_n$ converges – say $\sum_{n=1}^{\infty} a_n = 10$

just for example

Then: $\lim_{n\to\infty} S_n = 10$

So, say $S_{1000} \approx 10$

big number

$S_{999} \approx 10$

Then: $\underbrace{S_{1000} - S_{999}}_{a_{1000}} \approx 10 - 10 = 0$

So: If $\sum_{n=1}^{\infty} a_n$ converges, then

$\lim_{n\to\infty} a_n = 0$

Divergence Test If $\lim_{n \to \infty} a_n \neq 0$,

then $\sum_{n=1}^{\infty} a_n$ DIVERGES

$$\lim_{n \to \infty} a_n$$

$\neq 0$

$\sum a_n$ DIVERGES

$= 0$

we need to look harder to decide whether $\sum a_n$ DIV or CONERGES (ex) $\sum_{n=1}^{\infty} \sin n = \sin(1) + \sin(2) + \sin(3) + \cdots$

$\lim_{n\to\infty} \sin n \neq 0$
(DNE)

So by Divergence Test, $\sum_{n=1}^{\infty} \sin n$ DIVERGES

(ex) $\sum_{n=1}^{\infty} \frac{n^2+1}{n}$

$= \frac{1+1}{1} + \frac{4+1}{2} + \frac{9+1}{3} + \cdots$

$\lim_{n\to\infty} \left(\frac{n^2+1}{n}\right) = \infty \neq 0$

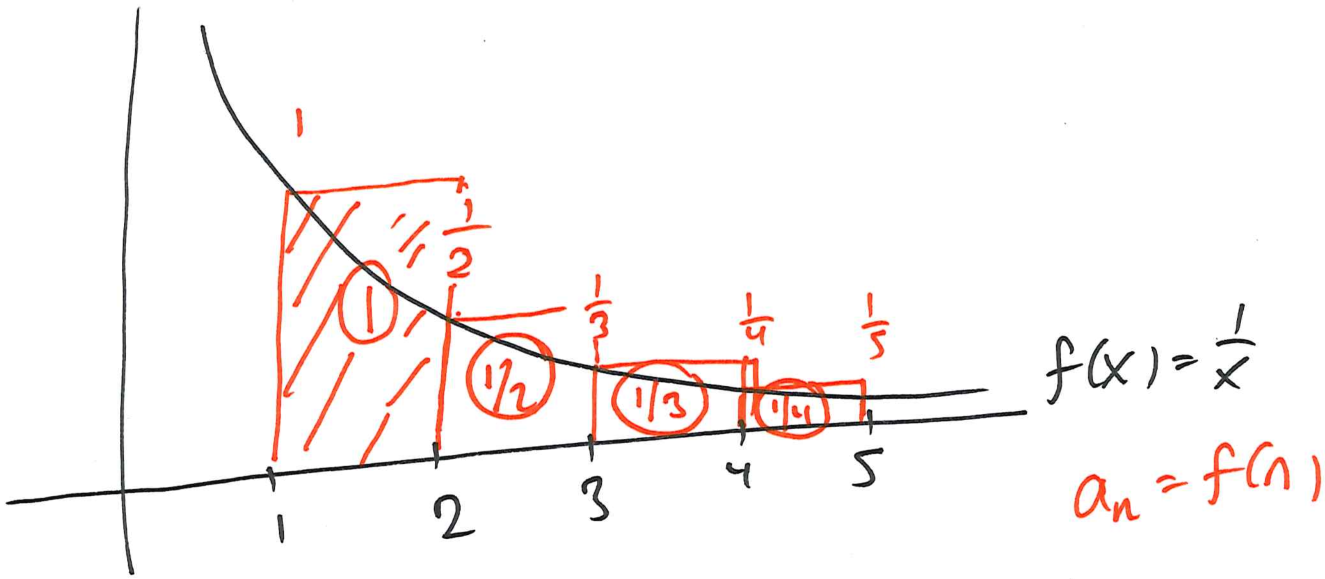
By Div Test, $\sum_{n=1}^{\infty} \frac{n^2+1}{n}$ DIVERGES (ex) $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

"Harmonic Series"

DIVERGES

Div or Conv ?

Try Div Test: $\lim_{n\to\infty} \frac{1}{n} = 0$ } Div Test tells us: try something else



$\sum_{n=1}^{\infty} \frac{1}{n}$ = area under rectangles

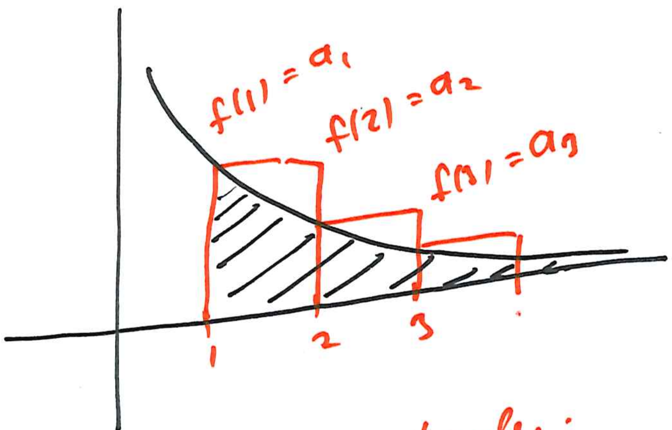
> area under $f(x)$ = $\int_1^{\infty} \frac{1}{x}\,dx = \infty$

So: $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (DIVERGES)

In General:

$$f(x): \begin{cases} \text{continuous} \\ \text{decreasing} \\ \text{positive} \end{cases}$$

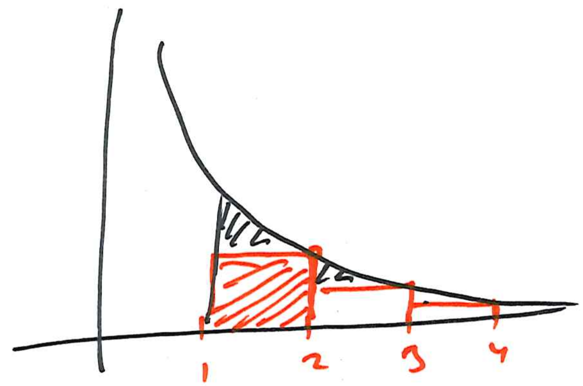
Sequence $a_n = f(n)$



Area under rectangles:

$a_1 + a_2 + a_3 + \cdots$

$$\sum_{n=1}^{\infty} a_n > \int_1^{\infty} f(x)\,dx$$



Area under rectangles:

$a_2 + a_3 + a_4 + \cdots$

$$= \sum_{n=2}^{\infty} a_n < \int_1^{\infty} f(x)\,dx$$

So:

$$\sum_{n=2}^{\infty} a_n \le \int_1^{\infty} f(x)\,dx \le \sum_{n=1}^{\infty} a_n$$

Integral Test

Let $f(x)$ be $\begin{cases} \text{continuous} \\ \text{decreasing} \\ \text{positive} \end{cases}$

Let $\{a_n\} = \{f(n)\}$

Then: $\int_1^\infty f(x)\,dx$ and $\sum_{n=1}^{\infty} a_n$ either

- both converge, or
- both diverge

ex $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n}$ Conv or Div?

Try Div Test:

$$\lim_{n\to\infty} \frac{1}{n \ln n} = 0$$

Div Test tells us: use something else

$$f(x) = \frac{1}{x \ln x}$$

- continuous
- decreasing (x large enough)
- positive

$$\int_3^{\infty} \frac{1}{x \ln x}\,dx \underset{\substack{u=\ln x \\ du=\frac{1}{x}dx}}{=} \int_{\ln 3}^{\infty} \frac{1}{u}\,du = \lim_{b\to\infty}\left[\ln|b| - \underbrace{\ln|\ln 3|}_{\text{some number}}\right)$$

$$= \infty$$

Integral diverges So: by Integral Test,

$\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n}$ DIVERGES as well.

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ Converge or Diverge?

Try Div Test:

$\displaystyle\lim_{n\to\infty} \frac{1}{n^2+1} = 0 \leftarrow$ (no help)

Let $f(x) = \dfrac{1}{x^2+1}$

- continuous
- positive
- decreasing

Justify int. test applies

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \left[\arctan b - \arctan 1 \right]$$

$$= \pi/2 - \pi/4 = \pi/4 \quad \underline{\text{CONVERGES}}$$

So, by $\boxed{\text{Integral Test}}$, $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1} \text{ converges}}$ as well.

name

(I don't know the sum!)

ex $\sum_{n=1}^{\infty} \frac{1}{n^p}$ Conv or div? ($p>0$ constant)

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p\le 1 \end{cases}$$

p-test

$\frac{1}{x^p}$:
- positive
- decreasing
- continuous

By integral tests

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p\le 1 \end{cases}$$

p-test for series

Ch 8.5 : Ratio & Comparison Tests

Still trying to answer: does $\sum a_n$ converge or diverge?

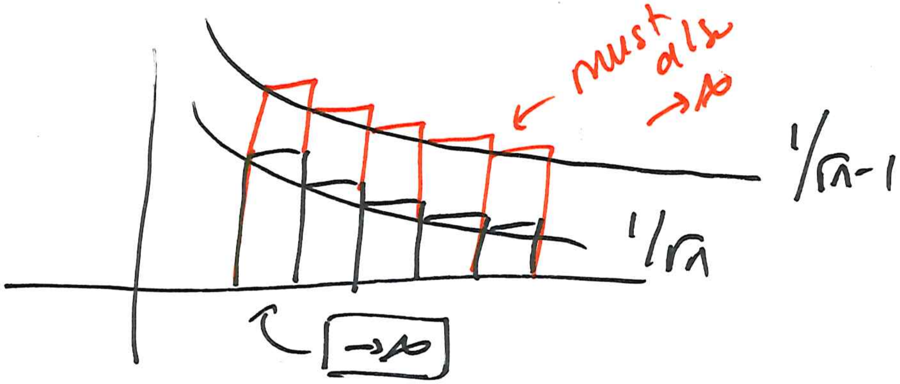
ex $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ Conv or Div?

Div Test: $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = 0$ no help

$\int$ test: $\displaystyle\int \frac{1}{\sqrt{x}-1}\,dx$ looks hard

Notice: $\sqrt{n}-1 < \sqrt{n}$

so $\displaystyle\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ DIVERGES by p-test ($p = 1/2$)



So: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ **DIVERGES**

Comparison Test

Let a_n, b_n be sequences with <u>positive</u> terms, and $a_n \le b_n$ for all large n.

① If $\sum_{n=1}^{\infty} a_n$ DIVERGES, then $\sum_{n=1}^{\infty} b_n$ DIVERGES as well

② If $\sum_{n=1}^{\infty} b_n$ CONVERGES, then $\sum_{n=1}^{\infty} a_n$ CONVERGES as well ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ conv or div?

Guess: maybe it behaves like $\left.\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}\right\}$ CONVERGES (p-test)

Notice: $\dfrac{1}{n^2+n} < \dfrac{1}{n^2}$

So, by Comparison Test: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ also converges

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What if we had tried to compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$?

Note: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES "harmonic series"

But: $\dfrac{1}{n^2+n} < \dfrac{1}{n}$

Can't compare to $\frac{1}{n}$