Partial Sums

(ex) Here's a sequence:

$a_1 = \frac{1}{2}$

$a_2 = \frac{1}{4}$

$a_3 = \frac{1}{8}$

$\frac{1}{16}$

$\vdots$

$a_n = \frac{1}{2^n}$

$\frac{1}{2}$

$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

Sequence of partial sums

$S_1 = 1/2$

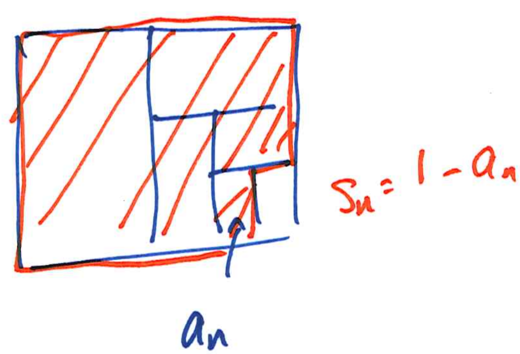
$S_2 = 3/4$

$S_3 = 7/8$

$S_4 = 15/16$

$\vdots$

$S_n = 1 - \frac{1}{2^n}$



(ex) $S_4 - S_3$

Long way: $S_4 = 15/16$, $S_3 = 7/8$

$S_4 - S_3 = 15/16 - 14/16 = 1/16$

$(S_4) - (S_3)$

$= (\cancel{a_1} + \cancel{a_2} + \cancel{a_3} + a_4) - (\cancel{a_1} + \cancel{a_2} + \cancel{a_3}) = a_4$

(ex) $S_{15} - S_{14} = a_{15} = \frac{1}{2^{15}}$

$$\left(\begin{array}{l} \cancel{a_1} \\ \cancel{+a_2} \\ \cancel{+a_3} \\ \vdots \\ \cancel{+a_{14}} \\ +a_{15} \end{array}\right) - \left(\begin{array}{l} \cancel{a_1} \\ \cancel{+a_2} \\ \cancel{+a_3} \\ \vdots \\ \cancel{+a_{14}} \end{array}\right)$$

In general,

$S_n - S_{n-1} = a_n$

(if $n \geq 2$)

(ex) (new sequence)

$\{a_n\}$, $\quad S_n = \sum_{k=1}^{n} a_k$

sequence of partial sums

Suppose $S_N = \frac{1}{2^N}$

Q: Are the terms a_n positive or negative?

$$\underset{\substack{\shortparallel \\ S_1}}{a_1} > S_2 > S_3 > S_4 \cdots$$

$+a_2$ $\quad +a_3$ $\quad +a_4$

Another way: $\quad S_N - S_{N-1} = a_N$ (as long as $N \geq 2$)

$$\frac{1}{2^N} - \frac{1}{2^{N-1}} = \frac{1}{2^N} - \frac{2}{2^N} = \frac{-1}{2^N} = a_N \text{ (negative)}$$

$a_1 = S_1 = \frac{1}{2}$ (positive)

Partial Sums of Geometric Sequence

$\{a_n\}: 1, r, r^2, r^3, r^4, r^5, \ldots$

$$S_N = 1 + r + r^2 + \cdots + r^N$$

$$S_{N+1} = \underbrace{1 + r + r^2 + \cdots + r^N}_{S_N} + r^{N+1} = S_N + r^{N+1}$$

sum ↑ ↑ terms of sequence

$$= 1 + r(\underbrace{1 + r + \cdots + r^{N-1} + r^N}_{S_N}) = 1 + rS_N$$

So: $S_N + r^{N+1} = 1 + rS_N$

$$S_N - rS_N = 1 - r^{N+1}$$

$$S_N(1-r) = 1 - r^{N+1}$$

$$\boxed{S_N = \frac{1 - r^{N+1}}{1-r}}$$

$$\underbrace{\sum_{k=0}^{N} r^k}_{= S_N} = \frac{1 - r^{N+1}}{1-r}$$

ex $\sum_{k=0}^{100} \left(\frac{5}{17}\right)^k = 1 + \left(\frac{5}{17}\right) + \left(\frac{5}{17}\right)^2 + \left(\frac{5}{17}\right)^3 + \dots + \left(\frac{5}{17}\right)^{100}$

geometric sum

$$= \frac{1 - (5/17)^{101}}{1 - 5/17}$$

ex $\sum_{k=1}^{100} 3^k = 3 + 3^2 + 3^3 + \dots + 3^{100}$

$$= [-1 + 1] + \underbrace{1 + 3 + 3^2 + 3^3 + \dots + 3^{100}}_{\text{formula}}$$

$$= -1 + \frac{1 - 3^{101}}{1 - 3}$$

$$= -1 + \frac{-1}{2}(1 - 3^{101}) = \frac{1}{2} \cdot 3^{101} - \frac{1}{2} - 1 = \frac{1}{2} \cdot 3^{101} - \frac{3}{2}$$

---

ex Suppose I know $100 + 100 = 200$

Then $99 + 100 = 200 - 1$

$-1 + 1 + 99 + 100 = 200 - 1$

$-1 + \boxed{100 + 100} = \boxed{200} - 1$

Ch. 8.3 : Infinite series

Definition:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} S_N$$

partial sum

ex)

$$1 + 0 + 0 + 0 \to 1$$

$$\sum_{n=0}^{\infty} (-1)^n = 1-1+1-1+1-1+1-1 \cdots$$

$$0 + 0 + 0 + 0 \to 0$$

The series diverges

$S_0 = 1$

$S_1 = 0$

$S_2 = 1$

$S_3 = 0$

$\vdots$

$$\lim_{n\to\infty} S_n = DNE$$

Before:

$$\sum_{k=0}^{N} r^k = 1 + r + r^2 + r^3 + \cdots + r^N = \frac{1-r^{N+1}}{1-r}$$

So:

$$\sum_{k=0}^{\infty} r^k = \lim_{N\to\infty} \sum_{k=0}^{N} r^k = \lim_{N\to\infty} \frac{1-r^{N+1}}{1-r} = \begin{cases} \frac{1}{1-r} & \text{if } |r|<1 \\ \text{DIV} & \text{if } |r|\geq 1 \end{cases}$$

$$\lim_{N\to\infty} r^{N+1} = \begin{cases} 0 & \text{if } |r|<1 \\ 1 & \text{if } r=1 \\ \text{DIV} & \text{if } |r|>1 \\ \text{DIV} & \text{if } r=-1 \end{cases}$$

ex $$\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = 1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots = \frac{1}{1-2/3}$$

$$= \frac{1}{1/3} = \boxed{3}$$

$$|r| = \left|\frac{2}{3}\right| < 1$$

ex $$\sum_{k=0}^{\infty} \left(-\frac{10}{9}\right)^k = 1 - \frac{10}{9} + \left(\frac{10}{9}\right)^2 - \left(\frac{10}{9}\right)^3 + \left(\frac{10}{9}\right)^4 - \left(\frac{10}{9}\right)^5 \dots$$

$\boxed{\text{DIVERGES}}$

$$|r| = \left|-\frac{10}{9}\right| = \frac{10}{9} > 1$$

ex $$\sum_{k=2}^{\infty} \frac{10^k}{9^{2k}} = \sum_{k=2}^{\infty} \frac{10^k}{(9^2)^k} = \sum_{k=2}^{\infty} \frac{10^k}{81^k} = \sum_{k=2}^{\infty} \left(\frac{10}{81}\right)^k$$

$$= \underbrace{\sum_{k=0}^{\infty} \left(\frac{10}{81}\right)^k}_{\text{formula}} - 1 - \frac{10}{81} = \boxed{\frac{1}{1-10/81} - 1 - \frac{10}{81}}$$

ex $\displaystyle\sum_{n=15}^{\infty} \frac{3^{n+1}}{4^n} = \sum_{n=15}^{\infty} \frac{3\cdot 3^n}{4^n} = 3\sum_{n=15}^{\infty} \left(\frac{3}{4}\right)^n$

$$= 3\left[\left(\frac{3}{4}\right)^{15} + \left(\frac{3}{4}\right)^{16} + \left(\frac{3}{4}\right)^{17} + \cdots\right]$$

$$= 3\left[\left(\frac{3}{4}\right)^{15}\left[\underbrace{1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots}_{\text{formula}}\right]\right]$$

$$= 3\left(\frac{3}{4}\right)^{15} \underbrace{\frac{1}{1-3/4}}_{\frac{1}{1-r}}$$