Theorem

Every monotone, bounded sequence converges.

If a sequence a_n has $\lim_{n\to\infty} a_n = L$, where L is a real #

eg $\{a_n\} = \frac{1}{2^n}$ $\qquad \lim_{n\to\infty} a_n = 0$ ← real #

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

so $\{\frac{1}{2^n}\}$ converges

We say a sequence $\{a_n\}$ is bounded if there exist real #s L, U such that $L \le a_n \le U$ for all a_n.

eg $\{a_n\} = \frac{1}{2^n}$

$0 \le a_n \le 1$

A sequence is <u>monotone</u> if

- it never decreases

or

- it never increases

$\{a_n\} = \{n\}$

e.g. Sequence : $1, 2, 3, 4, 5, 6, 7, \ldots$

monotone (never goes down)

$\{a_n\} = \{\frac{1}{n}\}$

Sequence : $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

monotone (never goes up)

$\{a_n\} = \{(-1)^{n+1}\}$

Sequence : $1, -1, 1, -1, 1, -1, \ldots$

$\searrow \nearrow$

<u>not</u> monotone

example: Jar of 100 pieces of candy.
You never re-fill.
After n days, a_n be # of pieces in jar.

$\{a_n\}$: sequence
$a_0 = 100$

- Bounded : $0 \le a_n \le 100$

- Monotone : $\{a_n\}$ never increases

By theorem: $\{a_n\}$ converges (we don't know what $\lim_{n\to\infty} a_n$ is, but it is a real number)

Possibility: candy is gross
100, 99, 99, 99, 99, 99, 99, ... $\lim_{n\to\infty} a_n = 99$

Another possibility: candy is OK
100, 80, 80, 80, 20, 20, 20, 0, 0, 0, 0, 0 ... $\lim_{n\to\infty} a_n = 0$

Geometric Sequences

Ratio between consecutive terms is constant.

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next to each other

always same

e.g. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

$\frac{1/4}{1/2} = \frac{1}{2}$ $\quad \frac{1/8}{1/4} = \frac{1}{2}$ $\quad \frac{1/16}{1/8} = \frac{1}{2}$ $\quad \frac{1/32}{1/16} = \frac{1}{2}$

geometric sequence

eg Suppose in 2017, avg size of a hard drive is 500 GB, increases 40% every year.

Recall:
$x + (40\% \text{ of } x)$
$= x + 0.4x = 1.4x$

$a_0 = 500$
$a_1 = (1.4)500$
$a_2 = (1.4)(1.4)500$
$a_3 = (1.4)(1.4)(1.4)500$

Recursive:
$a_0 = 500$
$a_n = (1.4)a_{n-1}$

Explicit:
$a_n = 500(1.4)^n$

Geometric Sequences have the form:

$$a_n = r^n a_0$$
$$= r(a_{n-1})$$

ex $\{a_n\} = \{(\frac{2}{3})^n\}$ $\qquad \lim_{n\to\infty} a_n = 0$

Sequence: $\frac{2}{3}, (\frac{2}{3})^2, (\frac{2}{3})^3, \ldots$

ex $\{a_n\} = \{1.1^n\}$ $\qquad \lim_{n\to\infty} a_n = \infty$

$1.1, (1.1)^2, (1.1)^3, (1.1)^4, \ldots$

In general: $a_n = r^n$: $\quad \lim_{n\to\infty} a_n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{DIV} & \text{if } |r| > 1 \end{cases}$

$\{a_n\} = \{1^n\}$

$1, 1, 1, 1, 1, \ldots$

Sequence of Partial Sums

Given a sequence a_n,

the sequence of partial sums S_n is:

$$S_n = a_1 + a_2 + \cdots + a_n$$

ex

some sequence a_n		add first n terms	sequence of partial sums. S_n
		$1/2$	
$a_1 =$	$1/2$		$1/2 = S_1$
$a_2 =$	$1/4$	$1/2 + 1/4 = 3/4$	$3/4 = S_2$
$a_3 =$	$1/8$	$1/2 + 1/4 + 1/8 = 7/8$	$7/8 = S_3$
	$1/16$	$1/2 + 1/4 + 1/8 + 1/16 = 15/16$	$15/16 = S_4$
	$\vdots$		
$a_n =$	$1/2^n$	$1/2 + 1/4 + 1/8 + \cdots + 1/2^n = 1 - \frac{1}{2^n}$	$1 - \frac{1}{2^n} = S_n$

$\lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} S_n = 1$

What is $S_4 - S_3$?

$$\frac{15}{16} - \frac{7}{8} = \frac{15}{16} - \frac{14}{16} = \frac{1}{16} = a_4$$

$$S_4 - S_3 = [\cancel{a_1} + \cancel{a_2} + \cancel{a_3} + a_4] - [\cancel{a_1} + \cancel{a_2} + \cancel{a_3}] = a_4$$

What is $S_{15} - S_{14}$?

$$S_{15} - S_{14} = a_{15}$$

(ex) $a_n = \frac{2}{10^n}$, $\quad S_N = \sum_{n=1}^{N} a_n$

$a_1 = \frac{2}{10} = 0.2$

$a_2 = \frac{2}{10^2} = 0.02$

$a_3 = \frac{2}{10^3} = 0.002$

$\vdots$

$S_1 = 0.2$

$S_2 = 0.22$

$S_3 = 0.222$

$\vdots$

$$\lim_{n\to\infty} S_n = 0.\overline{22} = 2/9$$

$$\lim_{n\to\infty} a_n = 0$$

(ex) Suppose $\{S_N\}$ is the partial sum of $\{a_n\}$

$$S_N = \frac{1}{2^N}$$

Q: are values a_n positive or negative?

S_N decreasing, so we must be adding negative #s

$$S_N - S_{N-1} = a_N$$

eg $S_4 - S_3 = a_4$: $a_4 = \frac{1}{2^4} - \frac{1}{2^3}$

$$(a_1 + a_2 + a_3 + a_4) - (a_1 + a_2 + a_3) = a_4$$

$$S_1 = a_1$$

So: $a_1 = S_1 = \frac{1}{2}$

After that, all a_n negative.

Geometric Partial Sums

$\{a_n\}_{n=0}^{\infty}$: $\quad 1, r, r^2, r^3, r^4, \ldots$

$$S_N = \sum_{n=0}^{N} a_n = (1 + r + r^2 + \cdots + r^N)$$

$$S_{N+1} = (\underbrace{1 + r + r^2 + \cdots + r^N}_{} + r^{N+1}) = \boxed{S_N + r^{N+1}}$$

$$S_{N+1} = (1 + r + r^2 + \cdots + r^N + r^{N+1})$$

$$= 1 + r(\underbrace{1 + r + \cdots + r^{N-1} + r^N}_{})$$

$$= \boxed{1 + r S_N}$$

So: $\quad S_N + r^{N+1} = 1 + r S_N$

$$S_N - r S_N = 1 - r^{N+1}$$

$$S_N (1 - r) = 1 - r^{N+1}$$

$$\boxed{S_N = \frac{1 - r^{N+1}}{1 - r}}$$

(ex) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}=\sum_{n=0}^{4} r^n$

$$=\frac{1-r^5}{1-r}=\frac{1-(1/3)^5}{1-1/3}=\frac{1-\frac{1}{3^5}}{2/3}$$

$$=\frac{3}{2}\left(1-\frac{1}{3^5}\right)$$

(ex) $\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\frac{2}{81}$

$$=\frac{2}{3}\left[1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}\right]=\frac{2}{3}\Big[\underbrace{(1/3)^0+(1/3)^1+(1/3)^2+(1/3)^3}_{\text{formula}}\Big]$$

$$=\frac{2}{3}\left(\frac{1-(1/3)^4}{1-1/3}\right)=\frac{\cancel{2}}{\cancel{3}}\left(\frac{1-\frac{1}{3^4}}{\cancel{2/3}}\right)$$

$$=1-\frac{1}{3^4}$$

Ch. 8.3 Infinite Series

$$\sum_{n=a}^{\infty} a_n = \lim_{N\to\infty} \underbrace{\sum_{n=a}^{N} a_n} = \lim_{N\to\infty} S_N$$

partial sum

ex $\{a_n\}_{n=0}^{\infty} = \{(-1)^n\}_{n=0}^{\infty}$

$1, -1, 1, -1, 1, -1, \dots$

$S_N = \sum_{k=0}^{N} (-1)^k = 1 - 1 + 1 - 1 + 1 \cdots$

$S_0 = 1$

$S_1 = 0$

$S_2 = 1$

$S_3 = 0$

$\lim_{n\to\infty} S_n$ DNE

So: $\sum_{n=0}^{\infty} a_n$ DIVERGES (limit DNE)

Geometric Series

$$\sum_{k=0}^{N} r^k = \frac{1-r^{N+1}}{1-r}$$ partial sum

$$\sum_{k=0}^{\infty} r^k = \lim_{N\to\infty}\left(\frac{1-r^{N+1}}{1-r}\right) = \begin{cases} \text{DIVERGE} & \text{if } |r| \geq 1 \\ \frac{1}{1-r} & \text{if } |r| < 1 \end{cases}$$

ex $$\underbrace{\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k}_{\text{geometric}} = \frac{1}{1-2/3} = \frac{1}{1/3} = 3$$

$$\left|\frac{2}{3}\right| < 1$$

ex $$\underbrace{\sum_{k=2}^{\infty} \left(\frac{2}{3}\right)^k}_{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots} = \underbrace{\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k}_{1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots} - 1 - \frac{2}{3} = 3 - 1 - \frac{2}{3} = 2 - \frac{2}{3} = \frac{4}{3}$$

ex $\displaystyle\sum_{n=2}^{\infty} \frac{10^n}{9^{2n}} = \sum_{n=2}^{\infty} \frac{10^n}{81^n} = \sum_{n=2}^{\infty} \left(\frac{10}{81}\right)^n = \sum_{n=0}^{\infty} \left(\frac{10}{81}\right)^n - 1 - \frac{10}{81}$

$\left|\frac{10}{81}\right| < 1$

$\displaystyle = \left(\frac{1}{1-10/81}\right) - 1 - \frac{10}{81} = \frac{1}{71/81} - \frac{91}{81} = \boxed{\frac{81}{71} - \frac{91}{81}}$

ex $\displaystyle\sum_{n=5}^{\infty} \frac{3^{n+1}}{2^{2n}} = \sum_{n=5}^{\infty} \frac{3\cdot 3^n}{4^n} = 3\sum_{n=5}^{\infty} \frac{3^n}{4^n} = 3\sum_{n=5}^{\infty} \left(\frac{3}{4}\right)^n$

$\left|\frac{3}{4}\right| < 1$

$\displaystyle = 3\left[\left(\frac{3}{4}\right)^5 + \left(\frac{3}{4}\right)^6 + \left(\frac{3}{4}\right)^7 + \cdots\right]$

$\displaystyle = 3\cdot\left(\frac{3}{4}\right)^5\cdot\left[\underbrace{1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \cdots}_{\text{formula}}\right]$

$\displaystyle = 3\left(\frac{3}{4}\right)^5 \underbrace{\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n} = 3\left(\frac{3}{4}\right)^5 \frac{1}{1-3/4}$

$\displaystyle = \frac{3^6}{4^5}\left(\frac{1}{1/4}\right) = \frac{3^6}{4^4}$

Usually: it's hard enough to say whether a series converges or diverges.

Note: Suppose $\sum_{n=1}^{\infty} a_n$ <u>converges</u> (say: $\sum_{n=1}^{\infty} a_n = 10$)

That means: $\lim_{N\to\infty} \underbrace{S_N}_{\text{partial sums}} = 10$

So: for really big N, $S_N \approx 10$

So: for really big N, $\underbrace{S_N - S_{N-1}}_{a_N} \approx 10 - 10 = 0$

sums

a_N ← things I'm adding

So: If $\sum_{n=1}^{\infty} a_n$ <u>converges</u>,

then $\lim_{n\to\infty} a_n = 0$