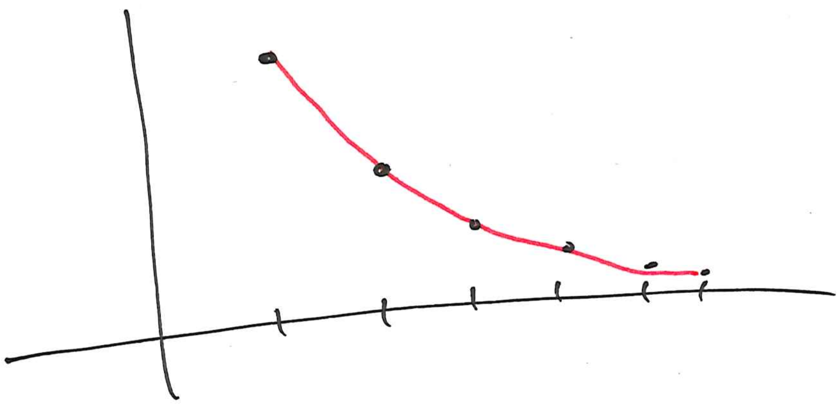
§ 8.2 : Sequences

ex $\{a_n\}_{n=1}^{\infty} = \{\frac{1}{n}\}_{n=1}^{\infty}$: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

$\lim_{n\to\infty} a_n = 0$



$f(x) = \frac{1}{x}$

domain $[1, \infty)$

function over a continuous domain

$\lim_{x\to\infty} \frac{1}{x} = 0$

Theorem | Suppose f is a function domain $[1,\infty)$ and $\lim_{x\to\infty} f(x) = L$, where L is a real #, or $\pm\infty$.

Let $\{a_n\}_{n=1}^{\infty} = \{f(n)\}_{n=1}^{\infty}$.

Then $\lim_{n\to\infty} a_n = L$

(ex) $\{a_n\} = \frac{n^2+1}{n^2+2}$

Sequence : $\frac{1+1}{1+2}, \frac{4+1}{4+2}, \frac{9+1}{9+2}, \ldots$

$\lim_{n\to\infty} a_n = 1$

$\lim_{x\to\infty} \frac{x^2+1}{x^2+2} = 1$

↑ understand: x is any real #

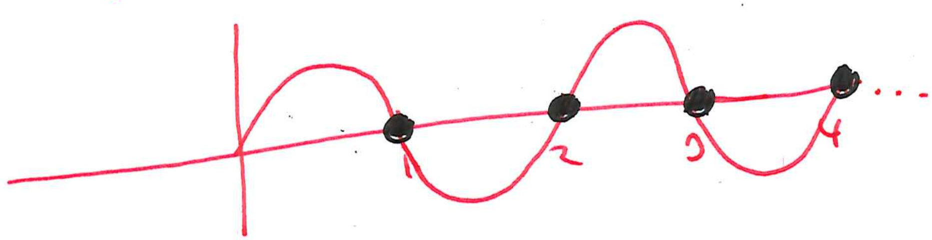
(ex) $\{a_n\} = \{\sin(\pi n)\}$

0, 0, 0, 0, 0, ...

$\lim_{n\to\infty} a_n = 0$

$f(x) = \sin(\pi x)$

$\lim_{x\to\infty} \sin(\pi x)$ DNE



$\{a_n\} = \{f(n)\}$

$\lim_{x\to\infty} f(x)$

exist

$\lim_{n\to\infty} a_n$ = same

not exist

we don't know what $\lim_{n\to\infty} a_n$ is

want looking more

Limit Laws for Sequences

Assume $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$

A, B exist

① $\lim_{n\to\infty} (a_n + b_n) = A + B$

② $\lim_{n\to\infty} c a_n = cA$, c : constant

③ $\lim_{n\to\infty} a_n b_n = AB$

④ $\lim_{n\to\infty} a_n / b_n = A/B$ if $B \neq 0$

(ex) $$\lim_{n\to\infty}\left(\underbrace{\frac{n^2+1}{n^2+2}}_{\to 1}+\underbrace{\sin(\pi n)}_{\to 0}\right)=1+0=1$$

(ex) $$\lim_{n\to\infty}\left(2\left(\underbrace{\frac{n^2+1}{n^2+2}}_{\to 1}\cdot\underbrace{\arctan n}_{\pi/2}\right)\right)=2(1)(\pi/2)=\pi$$

$\lim_{x\to\infty}\arctan x=\pi/2$

(ex) $$\lim_{n\to\infty}\left(\cos(\pi n)+\underbrace{\sin(\pi n)}_{0}\right)=\lim_{n\to\infty}\underbrace{\cos(\pi n)}_{-1,1,-1,1,-1,1,}$$ DNE

(ex) $$\lim_{n\to\infty}\left(\underbrace{\cos^2 n}_{DNE}+\underbrace{\sin^2 n}_{DNE}\right)=\lim_{n\to\infty}1=1$$

1, 1, 1, 1, 1

$a_n = \sin \pi n$

$\sin(\pi) = 0$
$\sin(2\pi) = 0$
$\sin(3\pi) = 0$
$\sin(4\pi) = 0$
$\vdots$

$b_n = \cos(\pi n)$

$n: 1, 2, 3, 4, \ldots$

$\cos(\pi) = -1$
$\cos(2\pi) = 1$
$\cos(3\pi) = -1$
$\cos(4\pi) = 1$

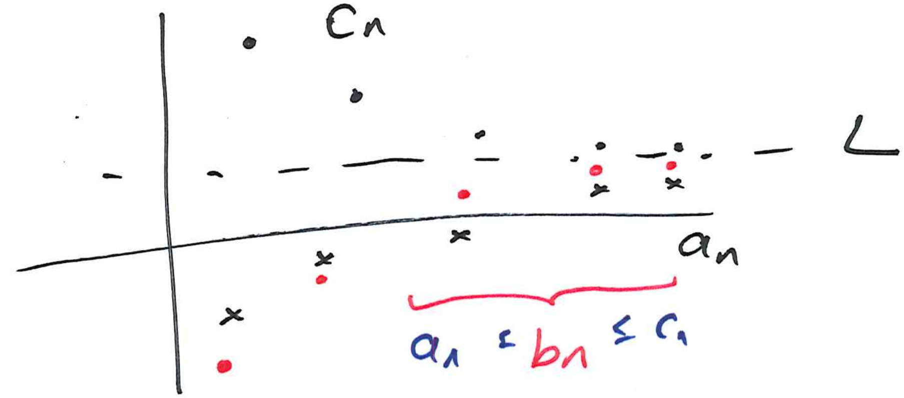
$$a_n = \underbrace{\underbrace{\cos(\pi n)}_{\text{limit DNE}} - \underbrace{\cos(\pi n)}_{\text{limit DNE}}}_{\text{limit } = 0} = 0$$

Squeeze Theorem for Sequences

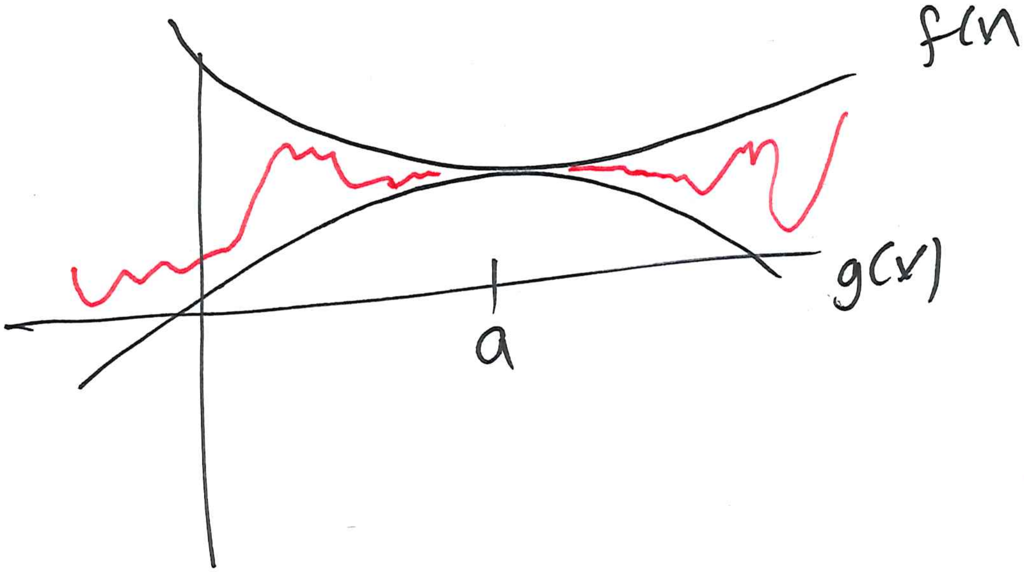
If • $a_n \le b_n \le c_n$ when $n \ge N$ for some N

<u>and</u> • $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$

Then $\lim\limits_{n\to\infty} b_n = L$ also



Functions:



If

• $g(x) \le h(x) \le f(x)$

and

• $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} f(x)$

Then $\lim\limits_{x\to a} h(x) = \lim\limits_{x\to a} g(x)$

(ex) $\lim_{n\to\infty} \frac{2n+\cos n}{n+1}$

$$\frac{2n\boxed{-1}}{n+1} \le \frac{2n+\boxed{\cos n}}{n+1} \le \frac{2n+\boxed{1}}{n+1}$$

$\lim_{n\to\infty}\left(\frac{2n-1}{n+1}\right) = \frac{2}{1} = 2$, $\lim_{n\to\infty} \frac{2n+1}{n+1} = \frac{2}{1} = 2$

So by Squeeze Theorem, $\lim_{n\to\infty} \frac{2n+\cos n}{n+1} = 2$.

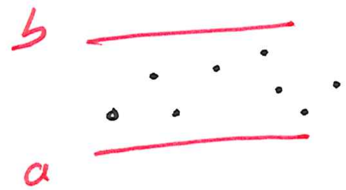
$\boxed{\text{Theorem}}$ Every bounded, monotone sequence converges.

A sequence converges if its limit ($n \to \infty$) exists is a real #

A sequence is bounded if there are some constants a, b such that all terms of the sequence lie between a & b

a "floor"

b "ceiling"



A sequence is <u>monotone</u> if:

- it never decreases

or

- it never increases

(staying the same is ok)

never increases

<u>monotone</u>

never decreases

<u>monotone</u>

sometimes ↗
sometimes ↘

<u>not monotone</u>

Ex Jar of candy never replenish.

a_n : # pieces on day n in jar

- bounded : ✓

$0 \le a_n \le$ (starting amount)

- monotone : ✓ a_n never increases

So: this sequence converge.

Geometric Sequences

Sequence with a common ratio between consecutive terms.

same divide

ex: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

$\frac{1}{4} / \frac{1}{2} = \frac{1}{2}$

$\frac{1/8}{1/4} = \frac{1}{2}$

$\frac{1/16}{1/8} = \frac{1}{2}$

ex) In 2017, avg hard drive holds 500 GB

Suppose storage capacity increases 40% each year.

$S_0 = 500$

$S_1 = 500 + 0.4(500)$
$= 1.4(500)$

$S_2 = \boxed{1.4(500)} + 0.4\left(\boxed{1.4 \cdot 500}\right) = (1.4 \cdot 500)(1 + 0.4) = (1.4)(500)(1.4)$
$= (1.4)(1.4)(500)$

In general: $S_n = 1.4\, S_{n-1}$ (recursive)

$S_n = (1.4)^n S_0$ (explicit)

Geometric sequence: $a_n = r\, a_{n-1}$

equivalently: $a_n = r^n a_0$