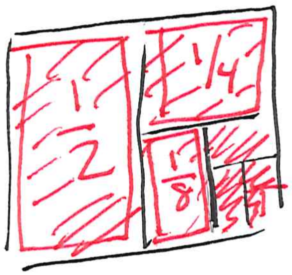
Sequences + Series

(ex)



Area of figure: 1

Areas of pieces:

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

list of #s (order)

"sequence"

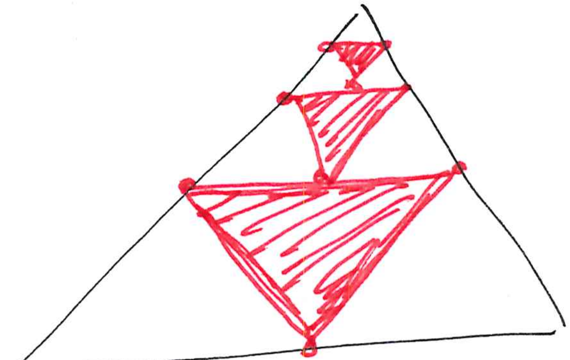
Sum areas of pieces:

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = 1$

"series"

- sum of terms in a sequence

(ex)



equilateral triangle

Area: 1

Area of Pieces:

$\frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \dots$

sequence

Sum of pieces:

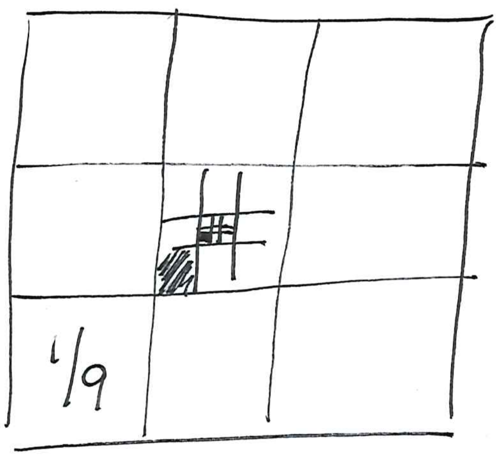
$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots = \frac{1}{3}$

series (sum)

ex

Sequence: $\frac{1}{9}, \frac{1}{9^2}, \frac{1}{9^3}, \frac{1}{9^4}, \dots$

Series: $\frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \frac{1}{9^4} + \dots = \frac{1}{8}$



We don't always want to use a sequence as a series

<u>Sequence</u> ordered list of numbers

A function whose domain is whole numbers

$f(1) = 3,689,257$

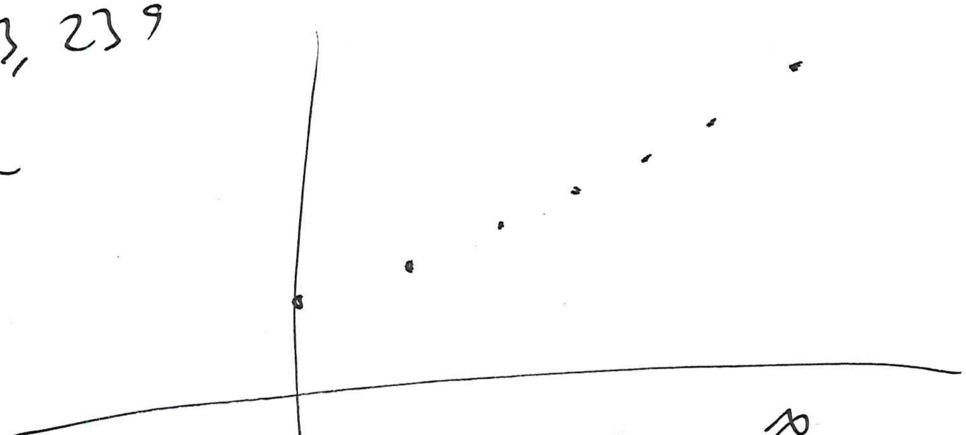
$f(2) = 4,324,810$

$f(3) = 4,833,239$

etc

$f(1/2)$ - no data

$f(-1)$ - no data



We write: $\{a_n\}_{n=1}^{\infty} = \{\frac{1}{2^n}\}_{n=1}^{\infty}$

each term ↗ ↖ rule

$a_1 = \frac{1}{2^1}, \quad a_2 = \frac{1}{2^2}, \quad a_3 = \frac{1}{2^3}$, etc

Recursive description

depends on previous terms

Explicit description

you can find a term without knowing what came before

ex $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \dots$

$$a_n = \frac{1}{2^n} \quad \text{explicit}$$

$$\left.\begin{aligned} a_1 &= \frac{1}{2} \\ \text{If } n>1, \quad a_n &= \frac{1}{2} a_{n-1} \end{aligned}\right\} \text{recursive}$$

a_n: term I want

a_{n-1}: term before

eg: $a_4 = \frac{1}{16}$

so: $a_5 = \frac{1}{2}\left(\frac{1}{16}\right) = \frac{1}{32}$

so: $a_6 = \frac{1}{2}\left(\frac{1}{32}\right) = \frac{1}{64}$

ex Fibonacci :

1, 1, 2, 3, 5, 8, 13, 21, ...

$F_0 \quad F_1 \quad F_2 \quad F_3 \quad F_4$

Rule (recursive)

$F_0 = 1$

$F_1 = 1$

if $n > 1$, $F_n = F_{n-1} + F_{n-2}$

$F_2 = F_1 + F_0 = 1 + 1 = 2$

$F_3 = F_2 + F_1 = 2 + 1 = 3$

ex $\frac{-1}{3}, \frac{1}{9}, \frac{-1}{27}, \frac{1}{81}, \ldots$

(explicit)

- $\{a_n\}_{n=1}^{\infty} = \{(\frac{-1}{3})^n\}_{n=1}^{\infty}$

(recursive)

$a_1 = \frac{-1}{3}$

- For $n>1$, $a_n = \frac{-1}{3} \cdot a_{n-1}$

$= \frac{1}{2+1}, \frac{1}{4+1}, \frac{1}{8+1}, \frac{1}{16+1}, \frac{1}{32+1}, \ldots$

ex $\frac{1}{3}, \frac{1}{5}, \frac{1}{9}, \frac{1}{17}, \frac{1}{33}, \ldots$

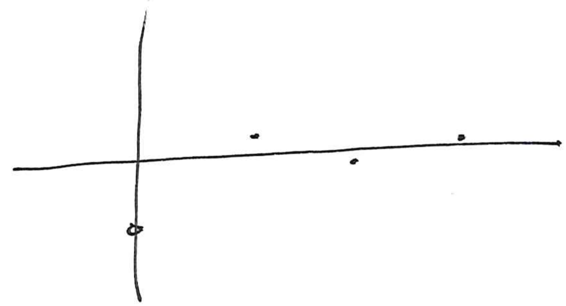
(explicit)

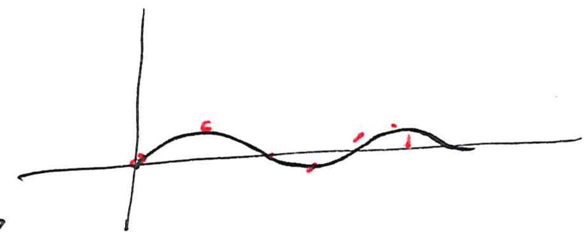
$\{a_n\}_{n=1}^{\infty} = \{\frac{1}{2^n+1}\}_{n=1}^{\infty}$

We can take limit as $n \to \infty$ of sequences

(ex) $a_n = \left(\frac{-1}{3}\right)^n$

$$\lim_{n \to \infty} a_n = 0$$





$\boxed{\text{Theorem}}$

Let $f(x)$ be a function, $\{a_n\}_{n=1}^{\infty} = \{f(n)\}_{n=1}^{\infty}$

If $\lim_{n \to \infty} f(n) = L$ for some real number L, then

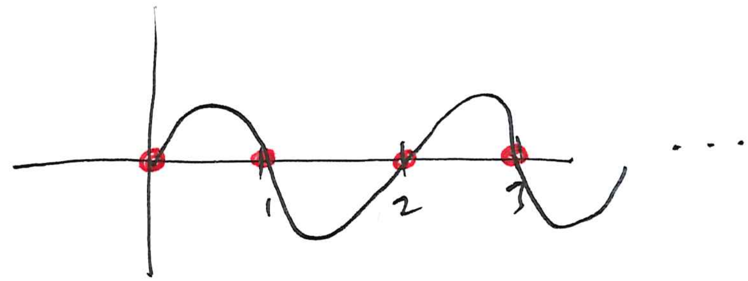
$$\lim_{n \to \infty} a_n = L$$

ex $a_n = \frac{n^2+1}{n^2+2}$ $\frac{1+1}{1+2}, \frac{4+1}{4+2}, \frac{9+1}{9+2}, \ldots$

$\lim_{n\to\infty} a_n = 1$

Let $f(x) = \frac{x^2+1}{x^2+2}$. Then $\lim_{x\to\infty} \frac{x^2+1}{x^2+2} = 1$

ex Let $f(x) = \sin(\pi x)$



$\lim_{x\to\infty} f(x) = DNE$

$a_n = f(n)$

So: $\lim_{n\to\infty} a_n = 0$

Sequence: $f(1), f(2), f(3), f(4), \ldots$

$\sin(\pi), \sin(2\pi), \sin(3\pi), \sin(4\pi), \ldots$

$0, 0, 0, 0$

function $f(x)$

sequence $a_n = f(n)$

$$\lim_{x \to \infty} f(x)$$

convergent
a real # L

$\lim_{n \to \infty} a_n = L$

(Theorem)

DNE

$\lim_{n \to \infty} a_n$ may or may not exist

(might cherry-pick values)

Limit Laws (sequences)

eg $$\lim_{n\to\infty}\left[\underbrace{\frac{n^2+1}{n^2+2}}_{\to 1} + \underbrace{\sin(\pi n)}_{\to 0}\right] = 1+1 = 1$$

Assume a_n, b_n are sequences, $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$,

A, B real #s

1. $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$

2. $\lim_{n\to\infty} c\,a_n = cA$, c constant

3. $\lim_{n\to\infty} a_n b_n = AB$

4. $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = A/B$ if $B \neq 0$

ex $\{a_n\} = \left\{\frac{n^2+3}{2n^2+1}\right\}$ $\qquad \lim_{n\to\infty} a_n = \frac{1}{2}$

$\{b_n\} = \{2\arctan n\}$ $\qquad \lim_{n\to\infty} b_n = \pi$

So: $c_n = \dfrac{2\arctan n\,(n^2+3)}{2n^2+1}$

(Limit Law #3) $\qquad \lim_{n\to\infty} c_n = \frac{1}{2}\cdot\pi = \pi/2$

ex $a_n = \frac{1}{n^2}$: $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$ $\qquad \lim_{n\to\infty} a_n = 0$

$b_n = 2n^2$: $2, 8, 18, 32, \ldots$ $\qquad \lim_{n\to\infty} b_n = \infty$

Can't simply say $\lim_{n\to\infty}(a_n b_n) = 0\cdot\infty \leftarrow ???$

$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty}\left(\frac{1}{\cancel{n^2}}\cdot 2\cancel{n^2}\right) = \lim_{n\to\infty}(2) = 2$

ex $\quad \lim_{n\to\infty} \left[\underbrace{\cancel{\sin(\pi n)}}_{0} + \underbrace{\cos(\pi n)}_{-1,\,1,\,-1,\,1,\,-1,\,1}\right]$ DNE

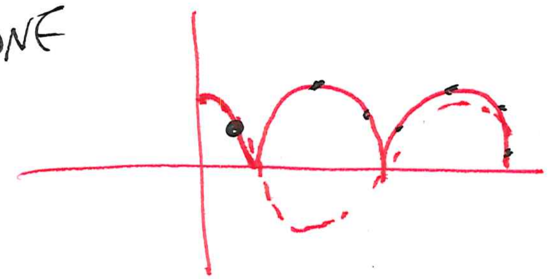
divergent sequence

Sequence: $-1, 1, -1, 1, -1, 1, -1, 1 \ldots$

ex $\quad \lim_{n\to\infty} \left[\cos^2(n) + \sin^2(n)\right] = \lim_{n\to\infty} [1] = 1$

$1, 1, 1, 1, 1, \ldots$

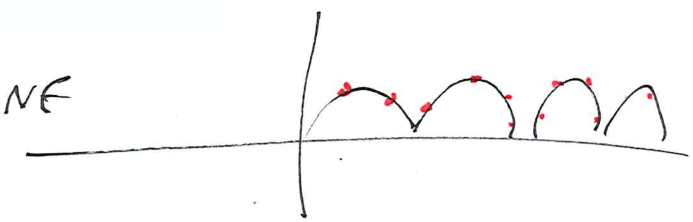
$\lim_{n\to\infty} \cos^2 n$: DNE



$\cos^2 x$

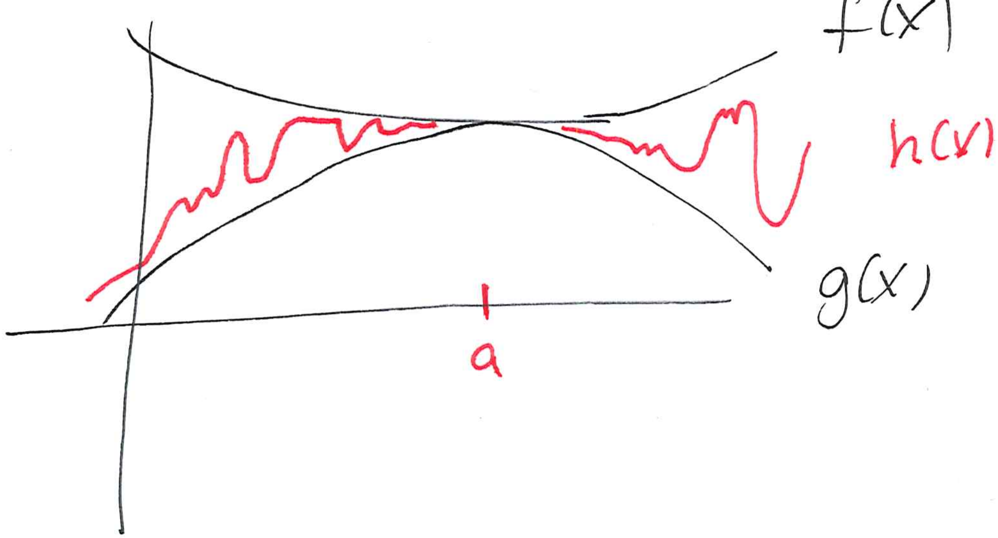
DIV

$\lim_{n\to\infty} \sin^2 n$ DNE



$\sin^2 x$

Squeeze Theorem for Sequences



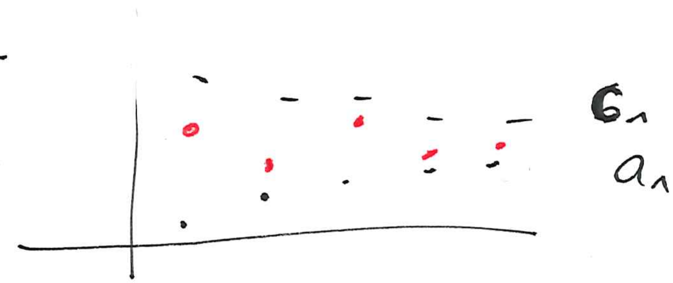
$g(x) \le h(x) \le f(x)$
for all x

If a_n, b_n, c_n are sequences, and:

- $a_n \le b_n \le c_n$ for all n larger than some value N

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

Then also $\lim_{n\to\infty} b_n = L$



ex $\{a_n\} = \left\{ \dfrac{2n + \cos n}{n+1} \right\}$

$$\underbrace{\frac{2n-1}{n+1}}_{b_n} \le \frac{2n+\cos n}{n+1} \le \underbrace{\frac{2n+1}{n+1}}_{c_n}$$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{2n+1}{n+1} = 2$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2n-1}{n+1} = 2$$

By Squeeze Theorem, $\lim_{n\to\infty} a_n = 2$.