Expected Value (Mean) of a Continuous Random Variable

Discrete:

eg 70% 70% 70% 80% 80%

Average: $\frac{70+70+70+80+80}{5} = \frac{3(70)+2(80)}{5}$

$= \frac{3}{5}(70) + \frac{2}{5}(80)$

Prob. value Prob value

Average: $\sum_{x} x \cdot Pr(X=x)$

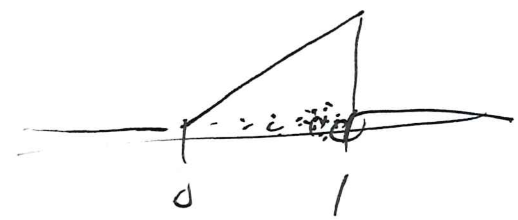
all possible values

If X is a continuous random variable with Probability Density Function $f(x)$, then the expected value ("expectation", "mean") of X is:

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

(ex) X is a cont. random variable with PDF:

$$f(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$



$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \cancel{\int_{-\infty}^{0} x \cdot 0\,dx}_{\,0} + \int_0^1 x \cdot 2x\,dx + \cancel{\int_1^{\infty} x \cdot 0\,dx}_{\,0}$$

$$= \int_0^1 2x^2\,dx = \frac{2}{3}x^3 \Big|_0^1 = \boxed{\frac{2}{3}}$$

(x) X is a cont. random variable with Cumulative Distribution Function:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 + \frac{3}{2}x & 0 \le x \le 1/2 \\ 1 & x > 1/2 \end{cases}$$

Definition of PDF: $F'(x)$ (when it exists)

$$f(x) = \frac{d}{dx}\{F(x)\} = \begin{cases} 0 & x < 0 \\ 2x + 3/2 & 0 < x < 1/2 \\ 0 & x > 1/2 \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \cancel{\int_{-\infty}^{0} x \cdot 0\,dx}_{0} + \int_{0}^{1/2} x \cdot (2x + 3/2)\,dx + \cancel{\int_{1/2}^{\infty} x \cdot 0\,dx}_{0}$$

$$\int_0^{1/2} 2x^2 + \frac{3}{2}x\,dx = \frac{2}{3}x^3 + \frac{3}{4}x^2 \Big|_0^{1/2} = \frac{2}{3}\cdot\frac{1}{8} + \frac{3}{4}\cdot\frac{1}{4} = \frac{2}{8\cdot 3} + \frac{3}{8\cdot 2}$$

$$= \frac{4}{8\cdot 6} + \frac{9}{8\cdot 6} = \frac{13}{48}$$

$\Rightarrow \frac{12}{48} = \frac{1}{4}$ (estimate)

Variance & Standard Deviation of a Continuous Random Variable

ex (discrete) ← for motivation

HW 1:	50%	50%	50%	50%	}	average: 0
average: 50						
value−avg:	0	0	0	0		
HW 2:	0%	0%	100%	100%		
average: 50						
value−average:	−50	−50	+50	+50	}	average: 0
$(\text{value}-\text{avg})^2$	50^2	50^2	50^2	50^2	}	average: 50^2 ← variance

fix it: $\sqrt{50^2} = 50$ ← standard deviation

Idea: measure "spread": how far people are from average

Want: average distance from average

→ we need to destroy signs

Continuous Case:

x - value

$E(X)$ - mean

$x - E(X)$ - distance from average

$(x - E(X))^2$ - destroy sign

want to average these

$$\boxed{\int_{-\infty}^{\infty} (x - E(X))^2 \cdot f(x)\, dx} = Var(X)$$

Variance of X (if $f(x)$ is the PDF of X)

Standard deviation of X: $\sigma(X) = \sqrt{Var(X)}$

sigma

Idea: "average distance from average"

$$Var(X) = \int_{-\infty}^{\infty} (x - \mathbb{E}(X))^2 \cdot f(x)\,dx$$

$$= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$ ← equivalent (sometimes easier)

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\int_{-\infty}^{\infty} x\, f(x)\,dx\right)^2$$

ex From before: PDF $f(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$

Already calculated: $\mathbb{E}(X) = \frac{2}{3}$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2 \cdot 2x\,dx = \int_0^1 2x^3\,dx = \frac{2}{4}x^4\Big|_0^1$$

$$= 1/2$$

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9}{18} - \frac{8}{18} = \boxed{\frac{1}{18}}$$

$$\sigma(X) = \sqrt{\frac{1}{18}} = \frac{1}{3\sqrt{2}}$$

(4) $\sigma(X) = \sqrt{Var(X)}$

What should you do if you find $Var(X) < 0$?

You should re-calculate variances always ≥ 0

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X: length of time used by students to finish a 1-hr exam.

PDF: $f(x) = \begin{cases} k(x^2+x) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{all other } x \end{cases}$

Recall: $Pr(a \leq X \leq b) = \int_a^b f(x)dx$

So: $\int_{-\infty}^{\infty} f(x)dx = Pr(-\infty \leq X \leq \infty) = 1$

(a) what is k? $1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^1 k(x^2+x)dx$

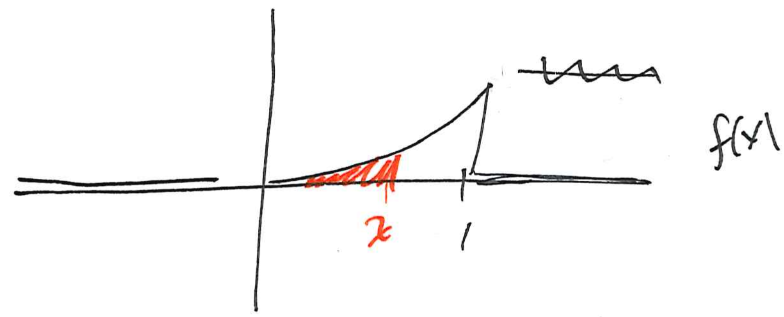
(b) Find CDF $F(x) = \int_{-\infty}^{x} f(t)dt$

$F(x) = Pr(X \leq x) = \int_{-\infty}^{x} f(t)dt$

(d) ~~Pr~~ what is prob a student finishes in less than half an hour?

(e), (f) $E(X)$, $Var(X)$, $\sigma(X)$

CDF:



$(k = 6/5)$

If $x<0$: $F(x)=0$

If $x>1$: $F(x)=1$

If $0 \le x \le 1$: $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$$= \int_0^x \frac{6}{5}(t^2+t)\,dt$$

$$= \frac{6}{5}\left(\frac{1}{3}t^3 + \frac{1}{2}t^2\right)\Big|_{t=0}^{t=x}$$

$$= \frac{6}{5}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)$$

$$F(x) = \begin{cases} 0 & \text{if } x<0 \\ \frac{6}{5}\left(\frac{1}{3}x^3+\frac{1}{2}x^2\right) & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x>1 \end{cases}$$

(a)

$Pr(X \le 1/2)$

$= F(1/2) =$

$\frac{6}{5}\left(\frac{1}{3}\cdot\frac{1}{8} + \frac{1}{2}\cdot\frac{1}{4}\right)$