Expected Value of a Continuous Random Variable

Discrete Case:

Homework: 70, 70, 70, 80, 80

Average: $\dfrac{70+70+70+80+80}{5} = \dfrac{3(70)+2(80)}{5} = \dfrac{3}{5}(70) + \dfrac{2}{5}(80)$

prob. that that value happened

value

value

Continuous Random Variable

$$\int x \cdot f(x)\, dx$$

add up

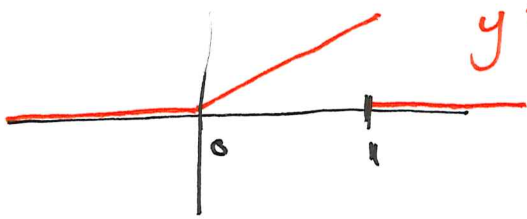
value

probability density

Def: The <u>expected value</u> ("expectation", "mean") of a continuous random variable X, with probability density function $f(x)$, is

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

(ex) Suppose X has PDF $f(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$

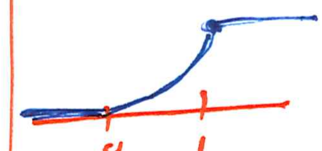


$$\int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_{-\infty}^{0} x \cdot \underbrace{0}_{\text{"}0} dx + \underbrace{\int_0^1 x \cdot 2x\, dx} + \int_1^{\infty} x \cdot \underbrace{0}_{\text{"}0} dx$$

$$= \int_0^1 2x^2\, dx = \frac{2}{3}x^3 \Big|_0^1 = \frac{2}{3} - 0 = \boxed{\frac{2}{3}} = \mathbb{E}(X)$$

$F(x)$ continuous

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$



CDF

(ex) X is a continuous random variable with Cumulative Distribution Function is

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 + \frac{3}{2}x & 0 \le x \le \frac{1}{2} \\ 1 & x > 1/2 \end{cases}$$

$F(1/2) = 1$:
$Pr(X \le 1/2) = 100\%$.
X always $\le 1/2$

What is $\mathbb{E}(X)$?

Recall:
$f(x) = F'(x)$
PDF

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$f(x) = \begin{cases} 0 & x < 0 \\ 2x + 3/2 & 0 < x < 1/2 \\ 0 & x > 1/2 \end{cases}$$

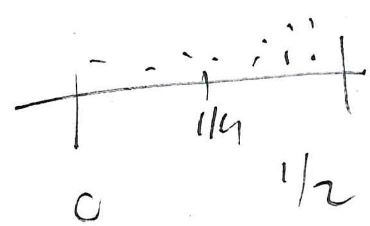
$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= 0 + \int_0^{1/2} x(2x + 3/2)\, dx + 0$$

$$= \int_0^{1/2} 2x^2 + \tfrac{3}{2}x\, dx = \tfrac{2}{3}x^3 + \tfrac{3}{4}x^2 \Big|_0^{1/2}$$

$$= \frac{2}{3} \cdot \frac{1}{2^3} + \frac{3}{4} \cdot \frac{1}{2^2} = \frac{1}{3 \cdot 2^2} + \frac{3}{2^4} = \frac{4}{3 \cdot 2^4} + \frac{9}{3 \cdot 2^4}$$

$$= \frac{13}{3 \cdot 16} = \frac{13}{48} > \frac{1}{4}$$



Variance & Standard Deviation

Motivation:

HW1 : 50 50 50 50 Avg: 50

HW2 : 0 0 100 100 Avg: $\frac{1}{2}(0) + \frac{1}{2}(100) = 50$

On average, how close is everyone to average?

Idea #1: Calculate avg of $(x - \underset{\text{avg}}{50})$

HW1:

grade	g-avg
50	0
50	0
50	0
50	0

Avg: 0

HW2:

grade	g-avg	$(g-\text{avg})^2$
0	-50	50^2
0	-50	50^2
100	+50	50^2
100	+50	50^2

Avg: 0 | Avg: 50^2

Idea #2

Fix it! $\sqrt{50^2} = 50$

X : continuous random variable

$\mathbb{E}(X)$: mean, expectation

$x - \mathbb{E}(X)$: how far value x is from $\mathbb{E}(X)$

$(x - \mathbb{E}(X))^2$: destroy +/−

↑ take Expectation

$$\int_{-\infty}^{\infty} (x-\mathbb{E}(X))^2 \cdot f(x)\, dx$$ — VARIANCE

$$\sqrt{\int_{-\infty}^{\infty} (x-\mathbb{E}(X))^2 \cdot f(x)\, dx}$$ — STANDARD DEVIATION

Def: The variance of a continuous random variable X, with probability density function $f(x)$, is:

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mathbb{E}(X))^2 \cdot f(x)\, dx$$

$$= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

rearranging

The standard deviation of X is

$$\sigma(X) = \sqrt{\mathrm{Var}(X)}$$

sigma

ex PDF from before:

$$f(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

We already calculated:

$$\mathbb{E}(X) = 2/3$$

Find: $Var(X)$, $\sigma(X)$

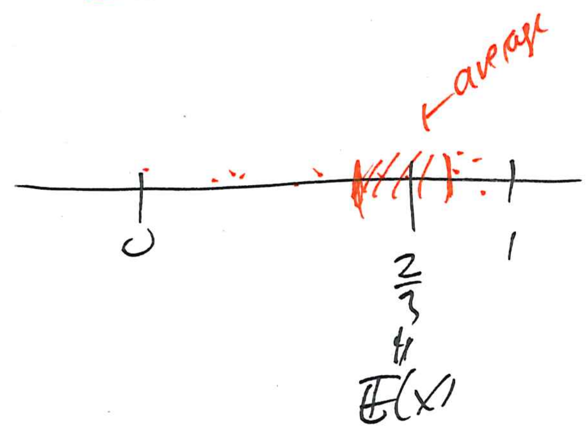
$$Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2 \cdot 2x\,dx = \int_0^1 2x^3\,dx = \frac{1}{2}x^4\Big|_0^1$$

$$= \frac{1}{2}$$

$$Var(X) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \boxed{\frac{1}{18}}$$

$$\sigma(X) = \sqrt{\frac{1}{18}} = \boxed{\frac{1}{3\sqrt{2}}}$$



Q: St dev is $\sqrt{\text{Variance}}$

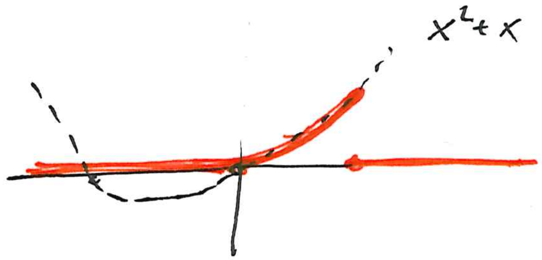
What do if $Var(X) < 0$? ← never happens!

$$Var(X) = \int_{-\infty}^{\infty} \underbrace{(x - E(X))^2}_{\geq 0} \cdot \underbrace{f(x)}_{\geq 0} \, dx$$

(ex) The length of time X used by students to complete a 1-hour exam is a random variable, with PDF:

p. A-82) in appendix

$$f(x) = \begin{cases} k(x^2+x) & \text{if } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$



Recall: $Pr(a \le X \le b) = \int_a^b f(x)\,dx$

So: $1 = \int_{-\infty}^{\infty} f(x)\,dx$

(a) what is k?

$1 = \int_{-\infty}^{\infty} f(x)\,dx$

$= \int_0^1 k(x^2+x)\,dx = k\left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)\Big|_0^1 = k\left(\frac{1}{3}+\frac{1}{2}\right) = k\left(\frac{5}{6}\right)$

$1 = k(5/6)$ $\boxed{k = 6/5}$

(b) Find the Cumulative Distribution Function

Recall: $F(x) = Pr(X \le x)$

$= Pr(-\infty \le X \le x)$

$= \int_{-\infty}^{x} f(t)dt$

$\int_{-\infty}$

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & \text{if } x < 0 \\ (\frac{1}{3}x^3 + \frac{1}{2}x^2)6/5 & 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$

(d) Find the probability that a randomly selected student will finish the exam in less than half an hour.

$$Pr(X \le \tfrac{1}{2}) = F(1/2) = \frac{6}{5}\left(\frac{1}{3}\cdot\frac{1}{2^3} + \frac{1}{2}\cdot\frac{1}{2^2}\right)$$

(e) Find the expected time to complete the exam.

$\mathbb{E}(X) =$

$$\int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^1 x \cdot \frac{6}{5}(x^2+x)\,dx$$

$$= \int_0^1 \frac{6}{5}(x^3+x^2)\,dx = \frac{6}{5}\left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right)\Big|_0^1$$

$$= \frac{6}{5}\left(\frac{1}{4}+\frac{1}{3}\right) = \frac{6}{5}\left(\frac{7}{12}\right) = \frac{7}{10}$$ (42 min)

(f) Find $Var(X)$, $\sigma(X)$

$$Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^1 x^2 \cdot \frac{6}{5}(x^2+x)\,dx$$

$$= \int_0^1 \frac{6}{5}(x^4+x^3)\,dx = \frac{6}{5}\left(\frac{1}{5}x^5 + \frac{1}{4}x^4\right)\Big|_0^1 = \frac{6}{5}\left(\frac{1}{5}+\frac{1}{4}\right)$$

$$= \frac{6}{5}\left(\frac{5+4}{20}\right) = \frac{6 \cdot 9}{100} = \frac{54}{100}$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

$$= \frac{54}{100} - \left(\frac{7}{10}\right)^2 = \frac{54-49}{100} = \frac{5}{100}$$

$$\sigma(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{\frac{5}{100}} = \boxed{\frac{\sqrt{5}}{10}}$$

standard deviation