Last Time

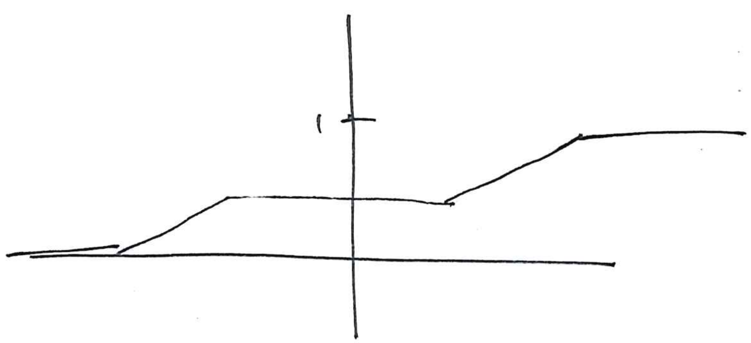
The cumulative distribution function for any random variable X, denoted by $F(X)$, is

$$F(x) = \Pr(X \leq x)$$

The CDF has the following properties:

- $0 \leq F(x) \leq 1$ for all values of x
- $\lim_{x \to -\infty} F(x) = 0$
- $\lim_{x \to \infty} F(x) = 1$
- $F(x)$ is a nondecreasing function of x

A random variable is <u>continuous</u> if its CDF is continuous.

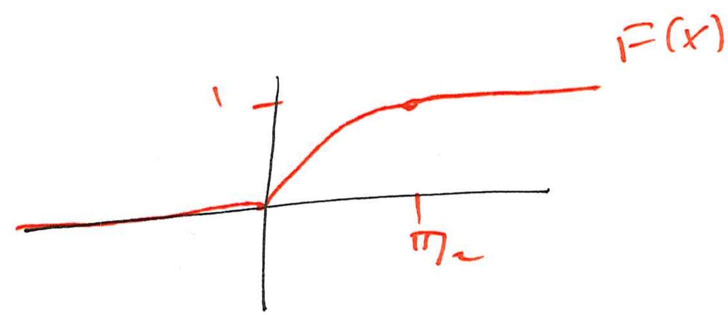


If X is a continuous random variable,
for any x, $Pr(X = x) = 0$

Then: $Pr(X \le x) = Pr(X < x)$

(ex) Suppose X is a random variable and its cumulative distribution function is

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sin x & \text{if } 0 \le x \le \pi/2 \\ 1 & \text{if } x > \pi/2 \end{cases}$$



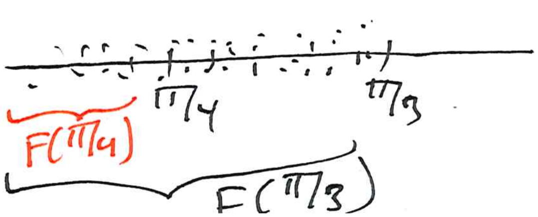
$Pr(X < 0) = F(0) = 0$

$Pr(X < 1) = F(1) = \sin(1)$

$Pr(\pi/4 \le X \le \pi/3) = F(\pi/3) - F(\pi/4)$

$= \sin(\pi/3) - \sin(\pi/4)$

$= \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} = \frac{\sqrt{3} - \sqrt{2}}{2}$

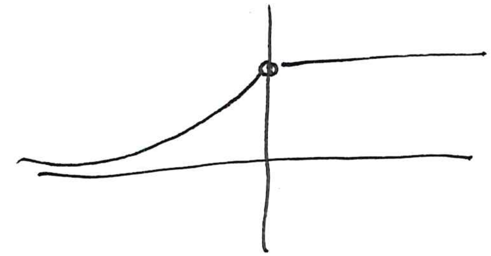


$Pr(X > \pi/4) = 1 - \underbrace{F(\pi/4)}_{Pr(X \le \pi/4)}$

$= 1 - \sin(\pi/4) = 1 - \frac{1}{\sqrt{2}}$

(ex) X is a random variable with Cumulative Distribution Function:

$$F(x) = \begin{cases} e^{x} & x \le 0 \\ 1 & x > 0 \end{cases}$$



$$Pr(X < 0) = F(0) = e^{0} = 1$$

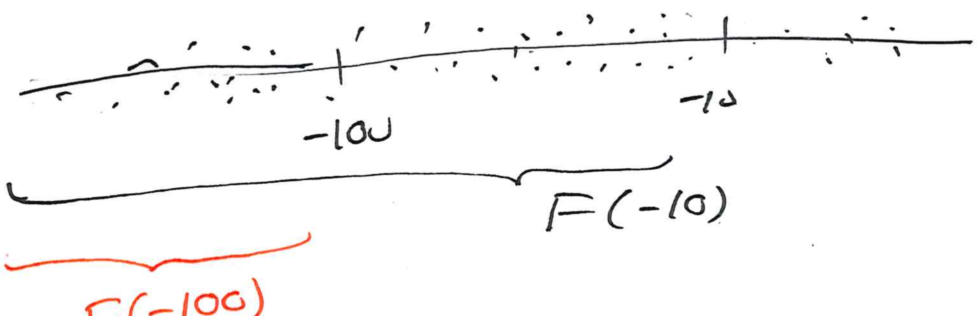
$$F(x) = Pr(X \le x) = Pr(X < x)$$

$$Pr(X < 1) = F(1) = 1$$

$$Pr\left(X > \tfrac{-1}{2}\right) = Pr\left(X \ \underline{\text{not}} \ X \le -1/2\right) = 1 - Pr(X \le -1/2)$$

$$= 1 - F(-1/2) = 1 - e^{-1/2} = 1 - \frac{1}{\sqrt{e}}$$

$$Pr(-100 < X < -10) = F(-10) - F(-100) = e^{-10} - e^{-100} = \frac{1}{e^{10}} - \frac{1}{e^{100}}$$

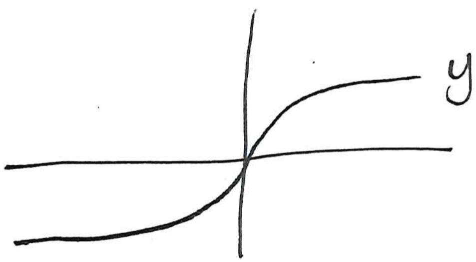


$$\Pr(X \geq x) = 1 - \Pr(X < x) = 1 - F(x)$$

$$\Pr(a \leq X \leq b) = F(b) - F(a)$$

ex $F(x) = k \arctan x + c$ for constant k, c

If F is a CDF, what are k, c?



$y = \arctan x$

$\lim_{x \to \infty} \arctan x = \pi/2$

- $\lim_{x \to \infty} F(x) = 1$

$\lim_{x \to \infty} k \arctan x + c = \boxed{k(\pi/2) + c = 1}$

- $\lim_{x \to -\infty} F(x) = 0$

$\lim_{x \to -\infty} k \arctan x + c = \boxed{k(-\pi/2) + c = 0}$

Find k, c

add 2 equations:

$2c = 1$, so $\boxed{c = 1/2}$

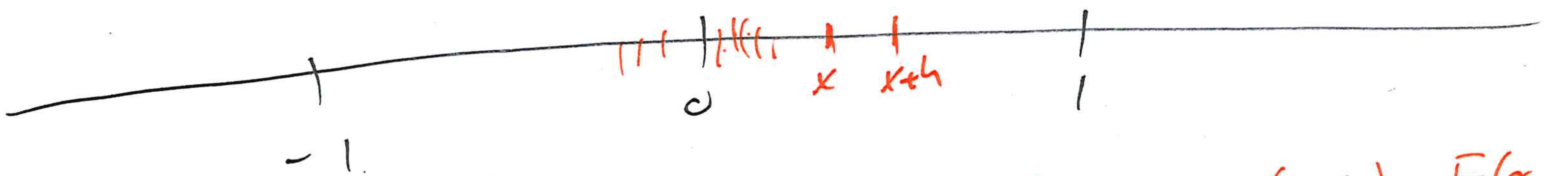
$k(\pi/2) + c = 1$

$k(\pi/2) = 1/2$ $\quad \boxed{k = 1/\pi}$

So $\boxed{F(x) = \frac{1}{\pi} \arctan x + \frac{1}{2}}$

Frustrating: $Pr(X=x)=0$ when X continuous random variable

Probability Density Function



$$Pr(X \approx x) : \frac{Pr(x \le X \le x+h)}{h} = \frac{F(x+h)-F(x)}{h} \approx F'(x)$$

Let $F(x)$ be the CDF of a continuous random variable X.

The PDF: Probability Density Function of X is

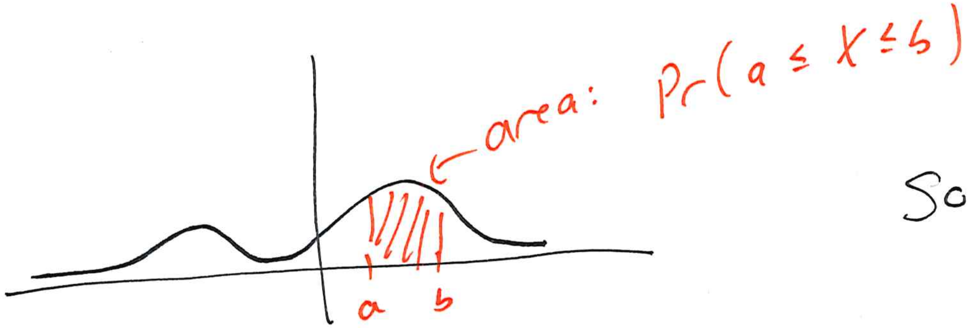
$$f(x) = \frac{d}{dx}\{F(x)\}$$ when it exists

We often go the "other way":

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$F(x)$: CDF; $f(t)$: PDF

$$Pr(a \le X \le b) = F(b) - F(a) = \int_a^b f(t)\,dt$$



So: $$\int_{-\infty}^{\infty} f(x)\,dx = Pr(-\infty < X < \infty) = 1$$

Also: $f(x) \ge 0$ for all x

(ex) $f(x) = ae^{-|x|}$ for some constant a

If $f(x)$ is a probability density function

what is a?

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} ae^{-|x|}\,dx$$

$$= \int_{-\infty}^{0} ae^{-|x|}\,dx + \int_{0}^{\infty} ae^{-|x|}\,dx$$

$$= \int_{-\infty}^{0} ae^{x}\,dx + \int_{0}^{\infty} ae^{-x}\,dx = \lim_{p \to -\infty} \int_{p}^{0} ae^{x}\,dx + \lim_{b \to \infty} \int_{0}^{b} ae^{-x}\,dx$$

$(-|x| = -(-x) = x)$

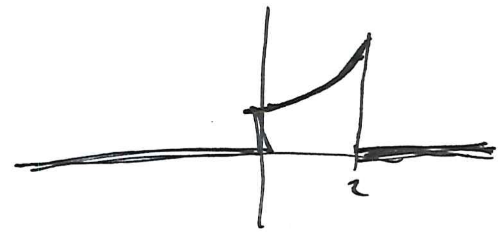
$$= \lim_{p \to -\infty} \left(ae^{x} \Big|_{p}^{0} \right) + \lim_{b \to \infty} \left(-ae^{-x} \Big|_{0}^{b} \right) = \left[ae^{0} - a(0) \right] + \left[0 - (-a \cdot e^{0}) \right]$$

$$= a + a = 2a = 1$$

So: $\boxed{a = 1/2}$

ex A-71 Let $f(x) = k(3x^2+1)$ for $0 \le x \le 2$, $f(x)=0$ elsewhere

$$f(x) = \begin{cases} k(3x^2+1) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$



i. Find k so that $f(x)$ is a PDF

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} k(3x^2+1)\,dx + \int_{2}^{\infty} 0\,dx$$

$$= \int_{0}^{2} k(3x^2+1)\,dx = 1$$

ii. $Pr(1 \le X \le 2) = \int_{1}^{2} k(3x^2+1)\,dx = \ldots$

iii. $F(x) = Pr(X \le x) = Pr(-\infty \le X \le x) = \int_{-\infty}^{x} f(t)\,dt$

iv. $Pr(X=1) = 0$