Def: A continuous random variable is one that has a continuous Cumulative Distribution Function (CDF)

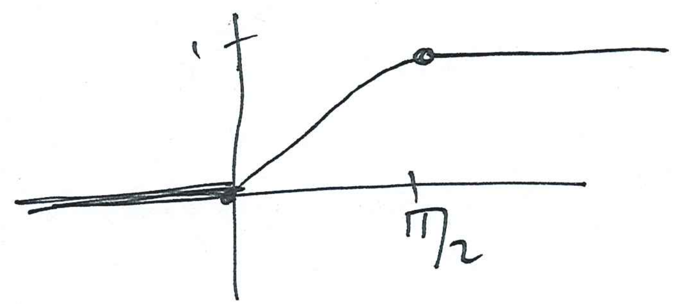
For any continuous random variable X,

$$\Pr(X = x) = 0$$

So: $\Pr(X \le x) = \Pr(X < x)$

(ex) Suppose X is a continuous random variable and its CDF is:

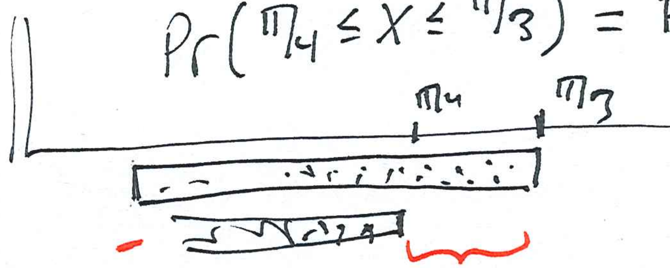
$$F(x) = \begin{cases} 0 & x < 0 \\ \sin x & 0 \le x \le \pi/2 \\ 1 & x > \pi/2 \end{cases}$$



$\Pr(X \le 0) = F(0) = 0$

$\Pr(X \le 1) = \sin(1)$

$$\Pr(\pi/4 \le X \le \pi/3) = F(\pi/3) - F(\pi/4)$$

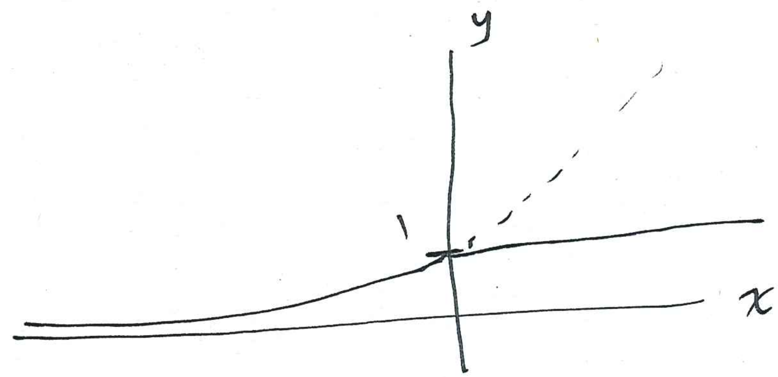


$= \sin(\pi/3) - \sin(\pi/4)$

$= \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} = \frac{\sqrt{3}-\sqrt{2}}{2}$

ex X is a random variable w/ cumulative distribution function

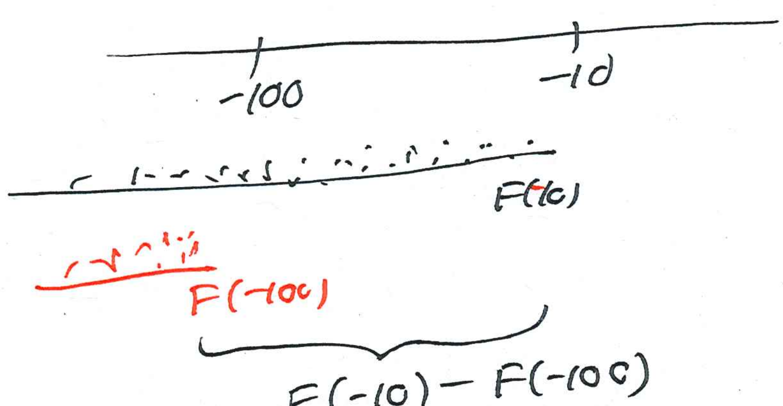
$$F(x) = \begin{cases} e^x & x \le 0 \\ 1 & x > 0 \end{cases}$$



$$Pr(X \le 0) = F(0) = e^0 = 1$$

$$Pr(X < 1) = Pr(X \le 1) = F(1) = 1$$

$$Pr(-100 \le X \le -10) = F(-10) - F(-100) = e^{-10} - e^{-100} = \frac{1}{e^{10}} - \frac{1}{e^{100}}$$



If $F(x)$ is a CDF:

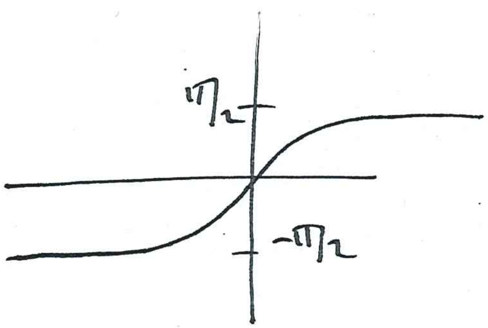
$$F(b) - F(a) = Pr(a \le X \le b)$$

$$Pr(X \ge a) = Pr(X \nleq a)$$

$$= 1 - F(a)$$

ex) Suppose $F(x) = k\arctan x + c$ for some constants k, c

If $F(x)$ is a Cumulative Distribution Function, what are k & c?



$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty}\left[k\arctan x + c\right]$$

$$= k(\pi/2) + c = 1$$

So: $\boxed{c = 1 - k(\pi/2)}$

$$\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty}\left[k\arctan x + c\right]$$

$$= k(-\pi/2) + c = 0$$

$$\boxed{c = k(\pi/2)}$$

$$1 - k(\pi/2) = k(\pi/2)$$

$$1 = 2k(\pi/2) = k\pi$$

$$\boxed{k = 1/\pi}$$

$$c = k(\pi/2) = \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2}$$

$$\boxed{c = 1/2}$$

Cumulative Distribution Function

T : temp outside (tomorrow)

Reasonable questions:

$F(0) = Pr(T \leq 0)$

$F(20) = Pr(T \leq 20)$

$F(50) = Pr(T \leq 50)$

Want a function

$F(x) = Pr(T \leq x)$

Properties of $F(x)$

$F(x)$ probability $\Rightarrow$ $0 \leq F(x) \leq 1$ for any x

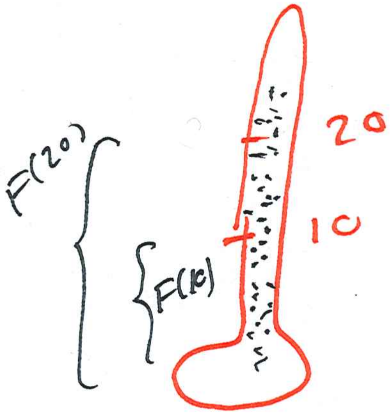
$F(1000) = Pr(T \leq 1000) = 1$

$F(-1000) = Pr(T \leq -1000) = 0$

$\lim_{x \to \infty} F(x) = 1$

$\lim_{x \to -\infty} F(x) = 0$

Compare $F(10)$, $F(20)$



$F(10) = Pr(T \le 10)$: proportion of days when $T \le 10$

$F(20) = Pr(T \le 20)$: days when $T \le 20$

So: $F(10) \le F(20)$

$F(x)$ is <u>nondecreasing</u>

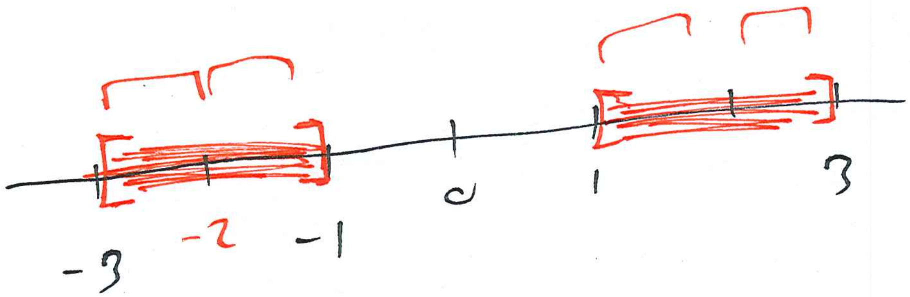
Def: The <u>cumulative distribution function</u> (CDF) of a random variable X is:

$$F(x) = Pr(X \le x)$$

- $0 \le F(x) \le 1$
- $\lim_{x \to \infty} F(x) = 1$
- $\lim_{x \to -\infty} F(x) = 0$
- $F(x)$ is a nondecreasing function of x

[eg $F(0) \le F(1) \le F(2) \le F(2.5) \ldots$]

ex



Choose any number from $[-3,-1] \cup [1,3]$.
"uniformly" (no preference), call this event X

$F(x)$: cumulative distribution function of X

So: $F(x) = Pr(X \le x)$

$Pr(X=1) = 0$

So: $Pr(X \le 1) = Pr(X < 1)$

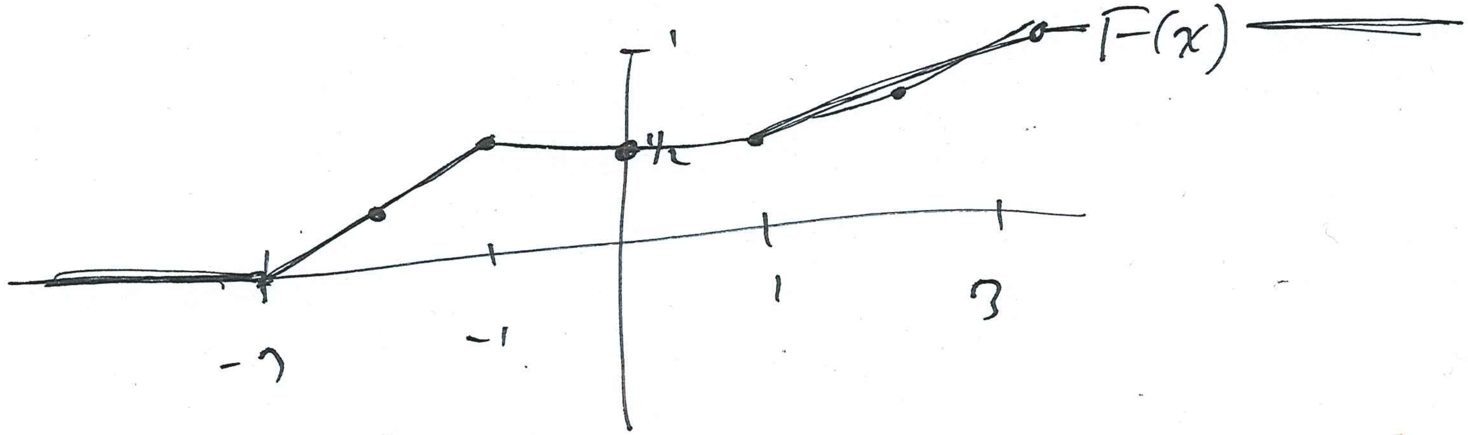
$F(0) = Pr(X \le 0) = \frac{1}{2}$

$F(-2) = Pr(X \le -2) = \frac{1}{4}$

$F(2) = Pr(X \le 2) = 3/4$

$F(-1) = 1/2$

$F(1) = 1/2$


F(x)
1
1/2
-2
-1
1
2

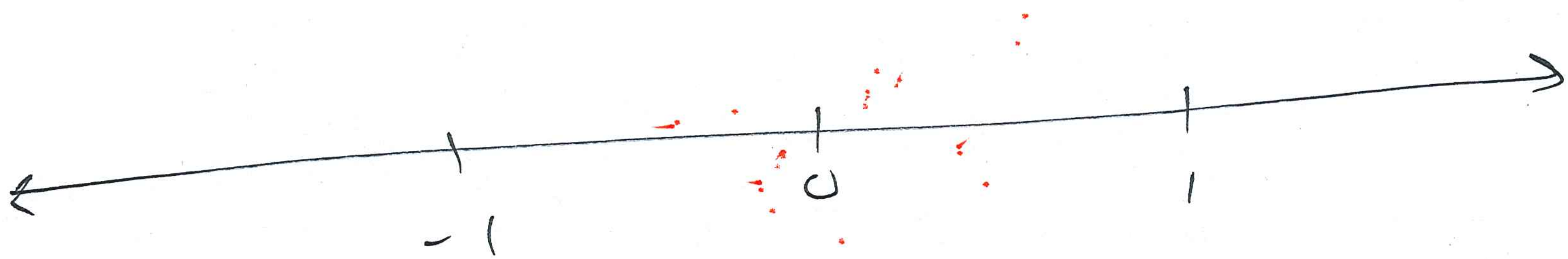
Probability Density Function

Problem: If X is a continuous random variable

then $Pr(X=x)=0$

But not all "regions" may be equally likely

So - how do we describe "preference?"



Thought!

$$Pr(X \approx x) \approx \frac{Pr(x \le X \le x+h)}{h} = \frac{F(x+h)-F(x)}{h}$$

derivative!

small h

Def:

Let $F(x)$ be the cumulative distribution function of a cont. random variable X.

The probability density function (PDF) of x

is:

$$f(x) = \frac{d}{dx}\{F(x)\}$$

(when it exists)

We often go the other way:

If $f(x)$ is the PDF to X,

then

$$\underbrace{F(x)}_{\text{CDF}} = \int_{-\infty}^{x} f(t)\,dt$$

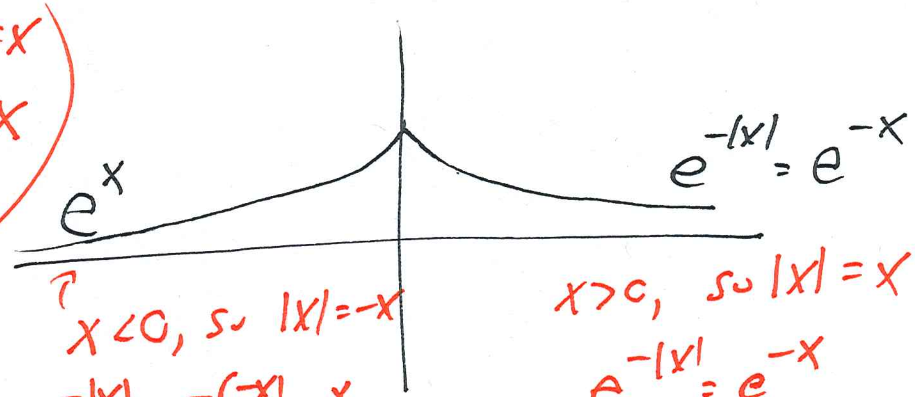
(ex) Suppose $f(x) = a e^{-|x|}$ is a PDF (a constant)

- What is a?
- What is $Pr(-3 \le X \le 1)$?

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

If $x \ge 0$, $|x| = x$

If $x < 0$, $|x| = -x$



$x < 0$, so $|x| = -x$

$e^{-|x|} = e^{-(-x)} = e^x$

$x > 0$, so $|x| = x$

$e^{-|x|} = e^{-x}$

$$\int_{-\infty}^{\infty} a \cdot e^{-|x|}\,dx$$

$$= \int_{-\infty}^{0} a \cdot e^{-|x|}\,dx + \int_{0}^{\infty} a \cdot e^{-|x|}\,dx$$

$$\overset{\text{(even)}}{=} 2\int_{0}^{\infty} a \cdot e^{-|x|}\,dx \underset{x \ge 0}{=} 2\int_{0}^{\infty} a \cdot e^{-x}\,dx = 2a \lim_{b\to\infty} \int_{0}^{b} e^{-x}\,dx$$

$$= 2a \lim_{b\to\infty}\left[-e^{-x} \Big|_{0}^{b} \right] = 2a\left[\lim_{b\to\infty} \left(-e^{-b} - -e^{-0}\right)\right]$$

$$= 2a \lim_{b\to\infty}\left[\underbrace{\frac{-1}{e^{b}}}_{0} + 1 \right] = 2a = 1$$

So: $\boxed{a = 1/2}$

$f(x)$: Prob. Density Function

$$F(b) - F(a) = \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt = \underbrace{\int_a^b f(t)\,dt}_{\text{useful!}}$$

$$\| $$

$$Pr(a \le X \le b)$$

So: $\int_a^b f(t)\,dt$ is some probability whenever $a < b$

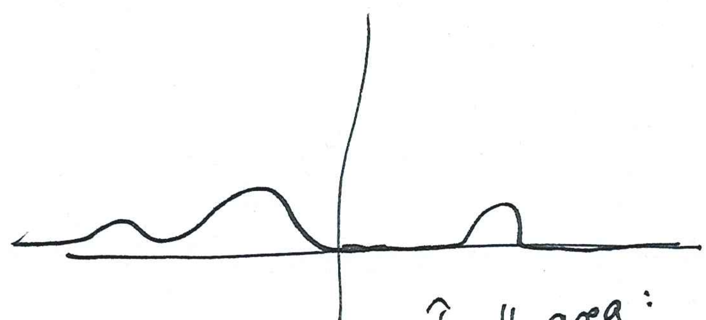
so: $0 \le \int_a^b f(t)\,dt$

So: $\boxed{f(x) \ge 0 \text{ for all } x}$

Also: $\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$

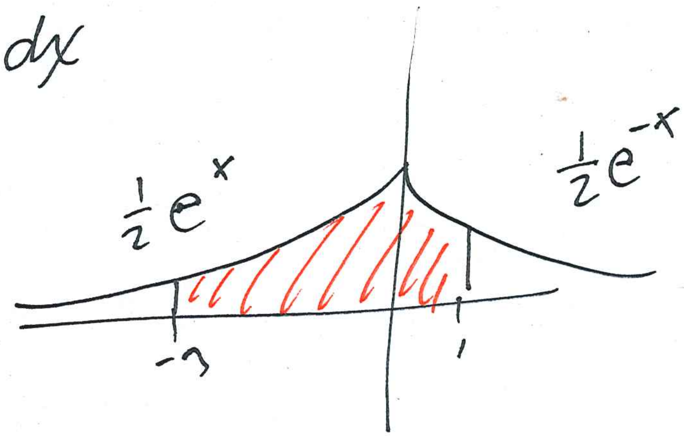
$$\| $$

$$Pr(-\infty \le X \le \infty)$$



all area: 1

$$\Pr(-3 \le x \le 1) = \int_{-3}^{1} f(x)\,dx$$



$$= \int_{-3}^{1} \frac{1}{2} e^{-|x|}\,dx$$

$$= \int_{-3}^{0} \frac{1}{2} e^{-|x|}\,dx + \int_{0}^{1} \frac{1}{2} e^{-|x|}\,dx$$

$$= \int_{-3}^{0} \frac{1}{2} e^{x}\,dx + \int_{0}^{1} \frac{1}{2} e^{-x}\,dx$$

$$= \left(\frac{1}{2}e^{x}\Big|_{-3}^{0}\right) + \left(\frac{-1}{2}e^{-x}\Big|_{0}^{1}\right)$$

$$= \left(\frac{1}{2}e^{0} - \frac{1}{2}e^{-3}\right) + \left(\frac{-1}{2}e^{-1} - \frac{-1}{2}e^{0}\right)$$

$$= \left(\frac{1}{2} - \frac{1}{2e^{3}}\right) + \left(\frac{-1}{2e} + \frac{1}{2}\right)$$

$$= \boxed{1 - \frac{1}{2e^{3}} - \frac{1}{2e}}$$

ex $$f(x) = \begin{cases} k(3x^2+1) & \text{if } 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

- Find the value of k that makes f a Probability Density Function.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$ ← solve for k

- $$Pr(1 \le X \le 2) = \int_1^2 f(x)\,dx$$ ← calculate

- Find the cumulative distribution function of X

$$F(x) = Pr(X \le x) = Pr(-\infty \le X \le x) = \int_{-\infty}^{x} f(t)\,dt$$ ↑ calculate

$$= \int_{-\infty}^{x} k(3t^2+1)\,dt$$

- Find the probability that $X = 1$. (0)

$$\int_1^1 f(x)\,dx = 0$$