Quiz #4 @ MLC

Last Time: p-test:

$\int_0^1 \frac{1}{x^p}\,dx$ DIVERGES if $p \geq 1$

CONVERGES if $p < 1$

$\int_1^\infty \frac{1}{x^p}\,dx$ DIVERGES if $p \leq 1$

CONVERGES if $p > 1$

ex

$$\int_0^1 \frac{1}{x^4}\,dx \longrightarrow$$

$p = 4 > 1$

DIVERGES

$$\int_0^1 \frac{1}{\sqrt[4]{x}}\,dx$$

$p = 1/4 < 1$

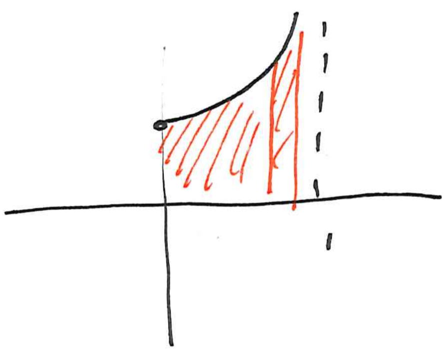
CONVERGES

← problem

ex

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-}\left[\int_0^b \frac{1}{\sqrt{1-x^2}}\,dx\right] = \lim_{b\to 1^-}\left[\arcsin(b) - \arcsin 0\right]$$

$$= \arcsin(1) = \pi/2$$



converges

Ch 7.9 Differential Equations

(ex) Some function $y(x)$, and

$$\frac{dy}{dx} + x^2 - 1 = y$$

$\cancel{(A)}$ $y = x^2+1 \longrightarrow (2x)+x^2-1 \overset{?}{=} (x^2+1) : x^2+2x-1 \overset{?}{=} x^2+1$ FALSE

✓ (B) $y = x^2+2x+1 \rightarrow (2x+2)+x^2-1 \overset{?}{=} x^2+2x+1 : x^2+2x+2-1 \overset{?}{=} x^2+2x+1$ TRUE

$\cancel{(C)}$ $y = -\frac{1}{3}x^3+x \rightarrow (-x^2+1)+x^2-1 \overset{?}{=} -\frac{1}{3}x^3+x , 0 = -\frac{1}{3}x^3+x$ FALSE

$y = x^2+2x+1$ is a solution to the differential equation

$$\frac{dy}{dx} + x^2 - 1 = y$$

ex $\frac{dy}{dx} = e^x$, $y(0) = 2$

what is y?

$$y = \int e^x \, dx$$

$$y = e^x + C$$

If $x = 0$, $y = 2$

$$2 = e^0 + C$$

$$C = 1$$

So: $\boxed{y = e^x + 1}$

ex $\quad y''(t) = 12t+1, \qquad y(0)=1, \qquad y(1)=10$

$$y'(t) = \int 12t+1 \, dt$$
$$= 6t^2+t+C$$

$$y(t) = \int 6t^2+t+C \, dt$$
$$= 2t^3+\frac{1}{2}t^2+Ct+D$$

If $t=0$, $y=1$

$1 = 0+D \qquad \boxed{D=1}$

If $t=1$, $y=10$

$10 = 2+\frac{1}{2}+C+D$

$10 = 2+\frac{1}{2}+C+1$

$\boxed{C=6.5}$

So: $\boxed{y(t) = 2t^3+\frac{1}{2}t^2+6.5t+1}$

Particular Example

$$y' = ky + b, \quad k, b \text{ constants}$$

Theorem: If $y' = ky + b$, then $\boxed{y = Ce^{kt} - b/k}$ for some constant C

ex) $y' = 3y + 7$, and $y(2) = 5$

k b

Theorem: $y = Ce^{3t} - 7/3$

Check: $\frac{dy}{dt} = 3y + 7$

$(C \cdot e^{3t} \cdot 3) \overset{?}{=} 3(Ce^{3t} - 7/3) + 7$

$3Ce^{3t} \overset{?}{=} 3Ce^{3t} \cancel{-7} \cancel{+7}$ TRUE

If $t = 2$, $y = 5$

$\frac{15}{3} = Ce^{6} - 7/3$

$\frac{22}{3} = Ce^{6}$

$C = \frac{22}{3 \cdot e^{6}}$

So: $y = \frac{22}{3 \cdot e^{6}} \cdot e^{3t} - 7/3$

$$\boxed{y = \frac{22}{3} e^{3t-6} - \frac{7}{3}}$$

ex $y' = \overset{k}{2}y + \overset{b}{8}$, $y(0) = 3$

$y = Ce^{2t} - 8/2$

$y = Ce^{2t} - 4$

If $t=0$, $y=3$

$3 = Ce^{2\cdot 0} - 4$

$3 = C - 4$

$C = 7$

So: $\boxed{y = 7e^{2t} - 4}$

Separable Differential Equations

ex $\quad \dfrac{dy}{dx} = y^2 x \qquad y(0) = 1$

long

$$\frac{1}{y^2} \cdot \frac{dy}{dx} = x$$

$$\int \frac{1}{y^2} \frac{dy}{dx}\, dx = \int x\, dx$$

$$y = f(x)$$

$$\frac{dy}{dx} = \frac{dy}{dx}$$

$$dy = \left(\frac{dy}{dx}\right) dx$$

$$\int \frac{1}{y^2}\, dy = \int x\, dx$$

short

$$\frac{1}{y^2}\, dy = x\, dx$$

$$\int \frac{1}{y^2} dy = \int x\, dx$$

$$\frac{-1}{y} = \frac{1}{2}x^2 + C$$

If $x=0$, $y=1$

$$\frac{-1}{1} = 0 + C$$

$$\boxed{C = -1}$$

Find C

$$\frac{-1}{y} + C_1 = \frac{1}{2}x^2 + C_2$$

$$\frac{-1}{y} = \frac{1}{2}x^2 + \underbrace{(C_2 - C_1)}$$

$$\frac{-1}{y} = \frac{1}{2}x^2 + C$$

$$\frac{-1}{y} = \frac{1}{2}x^2 - 1$$

solve for y

$$\frac{1}{y} = \frac{-1}{2}x^2 + 1$$

$$\boxed{y = \frac{1}{-\frac{1}{2}x^2 + 1}}$$

(ex) $\frac{dy}{dx} = e^{x-y}$ Find y.

$dy = e^{x-y} \cdot dx$

$dy = \frac{e^x}{e^y} dx$

$\int e^y dy = \int e^x dx$

$e^y = e^x + C$

$\boxed{y = \ln(e^x + C)}$

(ex) $\frac{dy}{dx} = y(4x^3-1)$, $y(0) = -2$

$\int \frac{1}{y}\,dy = \int (4x^3-1)\,dx$

$\ln|y| = x^4 - x + C$

Find C: If $x=0$, $y=-2$

$\ln|-2| = 0-0+C$

$\boxed{\ln 2 = C}$

$\ln|y| = x^4 - x + \ln 2$

$|y| = \underbrace{e^{x^4-x+\ln 2}}_{\text{POSITIVE}}$

$\boxed{y = -e^{x^4-x+\ln 2}}$

Recall:

$|y| = \begin{cases} y & \text{if } y \geq 0 \\ -y & \text{if } y < 0 \end{cases}$

2 possibilities:

$\cancel{y = e^{x^4-x+\ln 2}}$

or $-y = e^{x^4-x+\ln 2}$

Sugg. Prob 7.9