Error in Numerical Integration

Suppose you approximate $\int_1^2 \frac{1}{x}\,dx$ using different methods. Bound your error if you used $n=6$ intervals.

Note: $\ln 2 - \ln 1 = \ln 2$

$f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -x^{-2}$

$\boxed{f''(x) = 2x^{-3}}$ $\quad |f''(x)| = \left|\frac{2}{x^3}\right| \le \frac{2}{1^3} = 2$ $\quad \boxed{k=2}$

$+1 \le x \le 2$

$f'''(x) = -6x^{-4}$

$\boxed{f^{(4)}(x) = +24x^{-5}}$ $\quad |f^{(4)}(x)| = \left|\frac{24}{x^5}\right| \le \frac{24}{1^4} = 24$ $\quad \boxed{K=24}$

$1 \le x \le 2$

Midpoint: Formula sheet: $E_M \le \frac{k(b-a)^3}{24n^2} = \frac{2(2-1)^3}{24\cdot 6^2} = \frac{1}{12\cdot 6^2}$

Trapezoid: $E_T \le \frac{k(b-a)^3}{12n^2} = \frac{2(2-1)^3}{12\cdot 6^2} = \frac{1}{6\cdot 6^2}$

Simpson's: $E_s \le \frac{K(b-a)^5}{180 n^4} = \frac{24(2-1)^5}{180 \cdot 6^4} = \frac{24}{180 \cdot 6^4}$

formula sheet

Simpson's has best bound (smallest) on error

Fine Point:

$\frac{24}{180 \cdot 6^4} < \frac{1}{12 \cdot 6^2} < \frac{1}{6 \cdot 6^2}$

But not necessarily best approx (smallest error)

P: Prableen's Age
E: Elyse's Age

$P < 100$

$E < 50$

But $E > P$

Suppose you want to approximate $\int_1^2 \frac{1}{x}\,dx$

to within 10^{-4}, using midpoint rule.

Which n should I use?

$$E_M \leq \frac{k(b-a)^3}{24n^2} = \frac{2(b-a)^3}{24n^2} = \frac{2}{24n^2} = \frac{1}{12n^2} \leq \frac{1}{10^4}$$

find n

So:

$$12n^2 \geq 10^4$$

$$n^2 \geq \frac{10^4}{12}$$

$$n \geq \frac{10^2}{\sqrt{12}} \approx 28.8$$

Use $\boxed{n=29}$

$$\frac{1}{2} > \frac{1}{4}$$

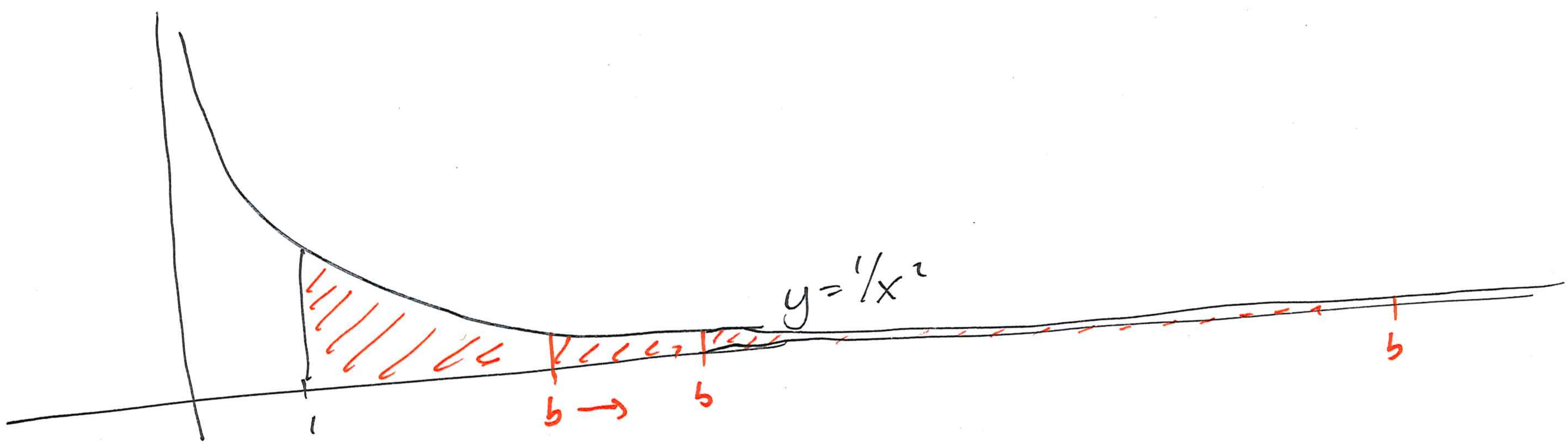
$$2 < 4$$

end Ch 7.7

Ch. 7.8 Improper Integrals

2 ways for a (definite) $\int$ to be improper

- unbounded interval, e.g. $\int_1^{\infty} \frac{1}{x^2}\,dx$
- unbounded function over interval e.g. $\int_{-1}^{+1} \frac{1}{x^2}\,dx$



ex $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\Big[\underbrace{\int_1^b \frac{1}{x^2}\,dx}_{\text{fits definition of definite integral}}\Big]$$

$$= \lim_{b\to\infty}\Big[\frac{-1}{x}\Big|_1^b\Big]$$

$$= \lim_{b\to\infty}\Big[\frac{-1}{b} - \frac{-1}{1}\Big] = \lim_{b\to\infty}\Big[1 - \cancel{\tfrac{1}{b}}^{\,0}\Big] = 1$$

ex $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty}\Big[\int_a^0 \frac{1}{1+x^2}\,dx\Big] + \lim_{b\to\infty}\Big[\int_0^b \frac{1}{1+x^2}\,dx\Big]$$

$$= \lim_{a\to-\infty}\big[\cancel{\arctan 0} - \arctan a\big] + \lim_{b\to\infty}\big[\arctan b - \cancel{\arctan 0}\big]$$

$$= -(-\pi/2) + \pi/2 = \pi$$

The integral converges (finite value)

ex $\displaystyle\int_{-\infty}^{\infty} \sin x\,dx = \int_{-\infty}^{0} \sin x\,dx + \int_{0}^{\infty} \sin x\,dx$

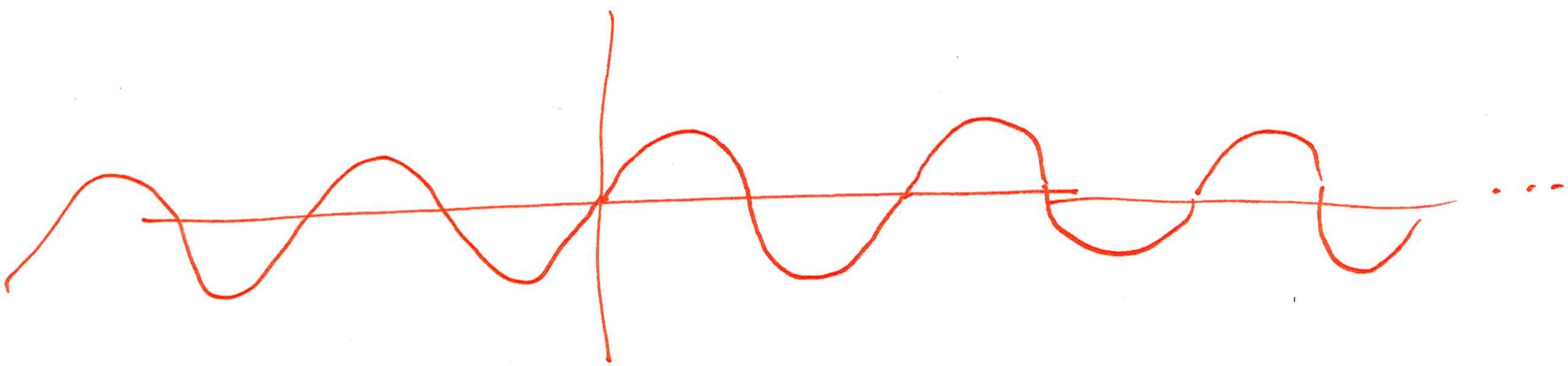
$$= \lim_{a\to-\infty}\left[\int_a^0 \sin x\,dx\right] + \lim_{b\to\infty}\left[\int_0^b \sin x\,dx\right]$$

$$= \lim_{a\to-\infty}\left[-\cos 0-(-\cos a)\right] + \lim_{b\to\infty}\left[-\cos b-(-\cos 0)\right]$$

$$= \lim_{a\to-\infty}\left[-1+\cos a\right] + \lim_{b\to\infty}\left[-\cos b+1\right]$$

DNE

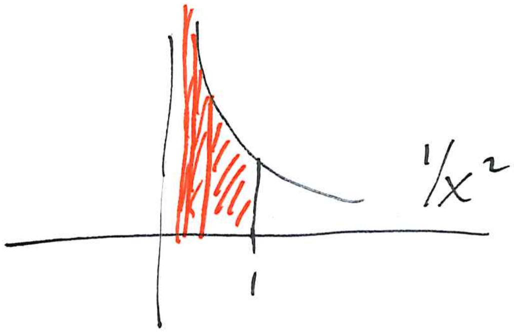
$\displaystyle\int_{-\infty}^{\infty} \sin x\,dx$ DIVERGES



ex $\displaystyle\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\left(\int_a^1 \frac{1}{x^2}\,dx\right) = \lim_{a\to 0^+}\left[\frac{-1}{x}\Big|_a^1\right]$

$$= \lim_{a\to 0^+}\left[\frac{-1}{1} - \frac{-1}{a}\right]$$

$$= \lim_{a\to 0^+}\left[\frac{1}{a} - 1\right] = \infty$$

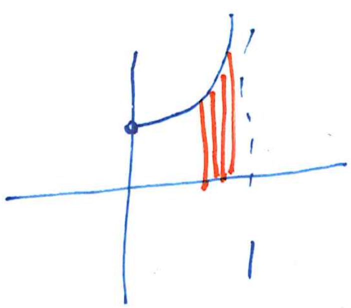


So $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$ DIVERGES

← problem bound

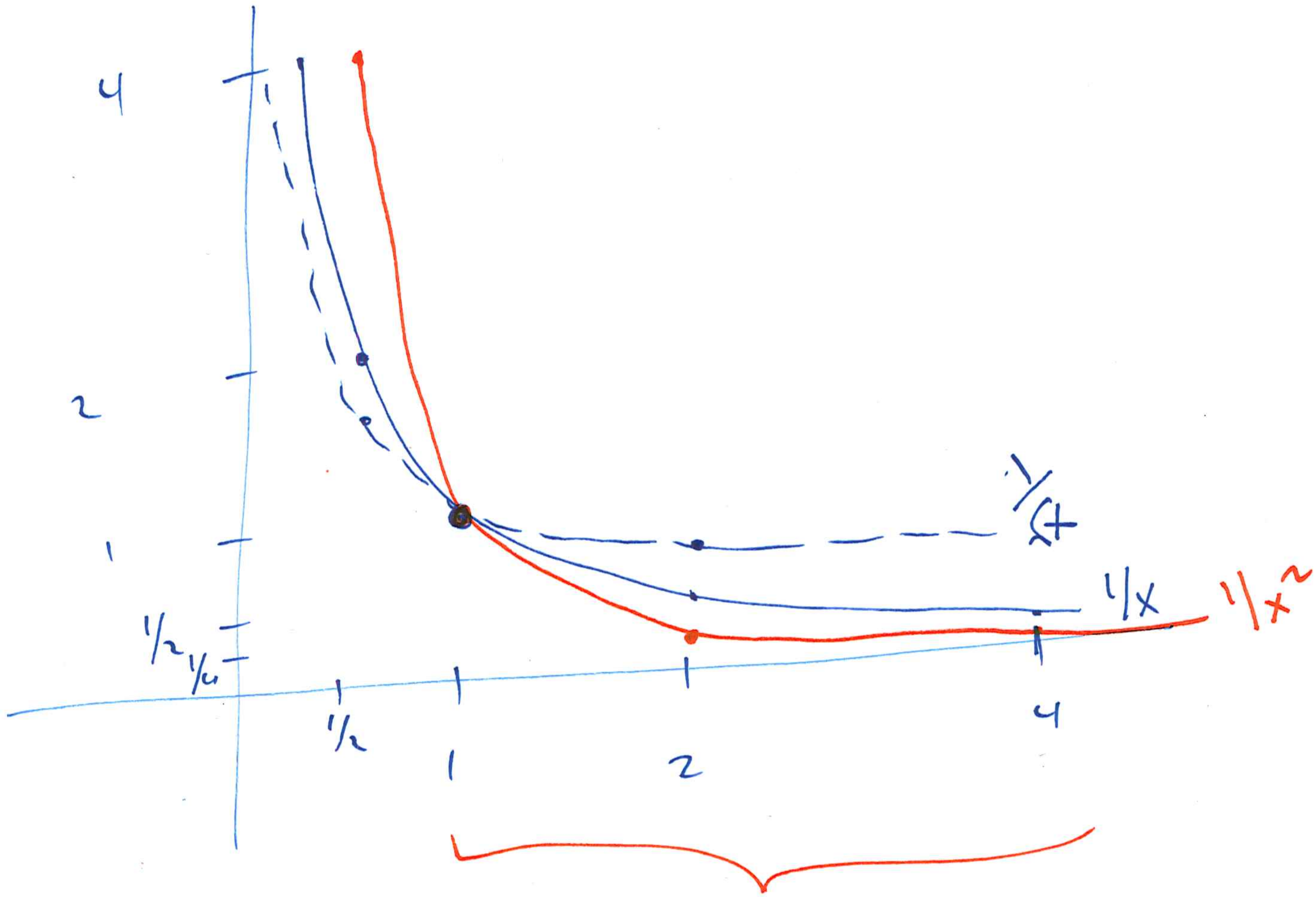
ex $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-}\left[\int_0^b \frac{1}{\sqrt{1-x^2}}\,dx\right] = \lim_{b\to 1^-}\left(\arcsin b - \cancel{\arcsin 0}\right)$

$$= \arcsin(1) = \boxed{\pi/2}$$



$\frac{1}{x^p}$, p positive #

$\frac{1}{(1/2)^2} = \frac{1}{1/4} = 4$



$\boxed{\text{p-test}}$

$\int_0^1 \frac{1}{x^p}\,dx$: converge if $p<1$

diverge if $p \geq 1$

$\int_1^\infty \frac{1}{x^p}\,dx$: converge if $p>1$

diverge if $p \leq 1$

$$\int_0^\infty \frac{1}{x^2}\,dx = \underbrace{\int_0^1 \frac{1}{x^2}\,dx}_{\substack{p=2>1 \\ \text{DIV}}} + \int_1^\infty \frac{1}{x^2}\,dx$$

DIVERGE