Improper Integrals

ex $$\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+}\left[\int_a^1 \frac{1}{x}\,dx\right] = \lim_{a\to 0^+}\left[\cancel{\ln 1}^{\,0} - \ln a\right] = \infty$$

DIVERGES

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\int_1^b \frac{1}{x}\,dx\right] = \lim_{b\to\infty}\left[\ln b - \cancel{\ln 1}^{\,0}\right] = \infty$$

DIVERGES

ex $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[\int_1^b x^{-2}\,dx\right] = \lim_{b\to\infty}\left[-x^{-1}\Big|_1^b\right]$$

CONVERGES

$$= \lim_{b\to\infty}\left[\frac{-1}{b} - \frac{-1}{1}\right] = \lim_{b\to\infty}\left[1 - \cancel{\frac{1}{b}}^{\,0}\right] = 1$$

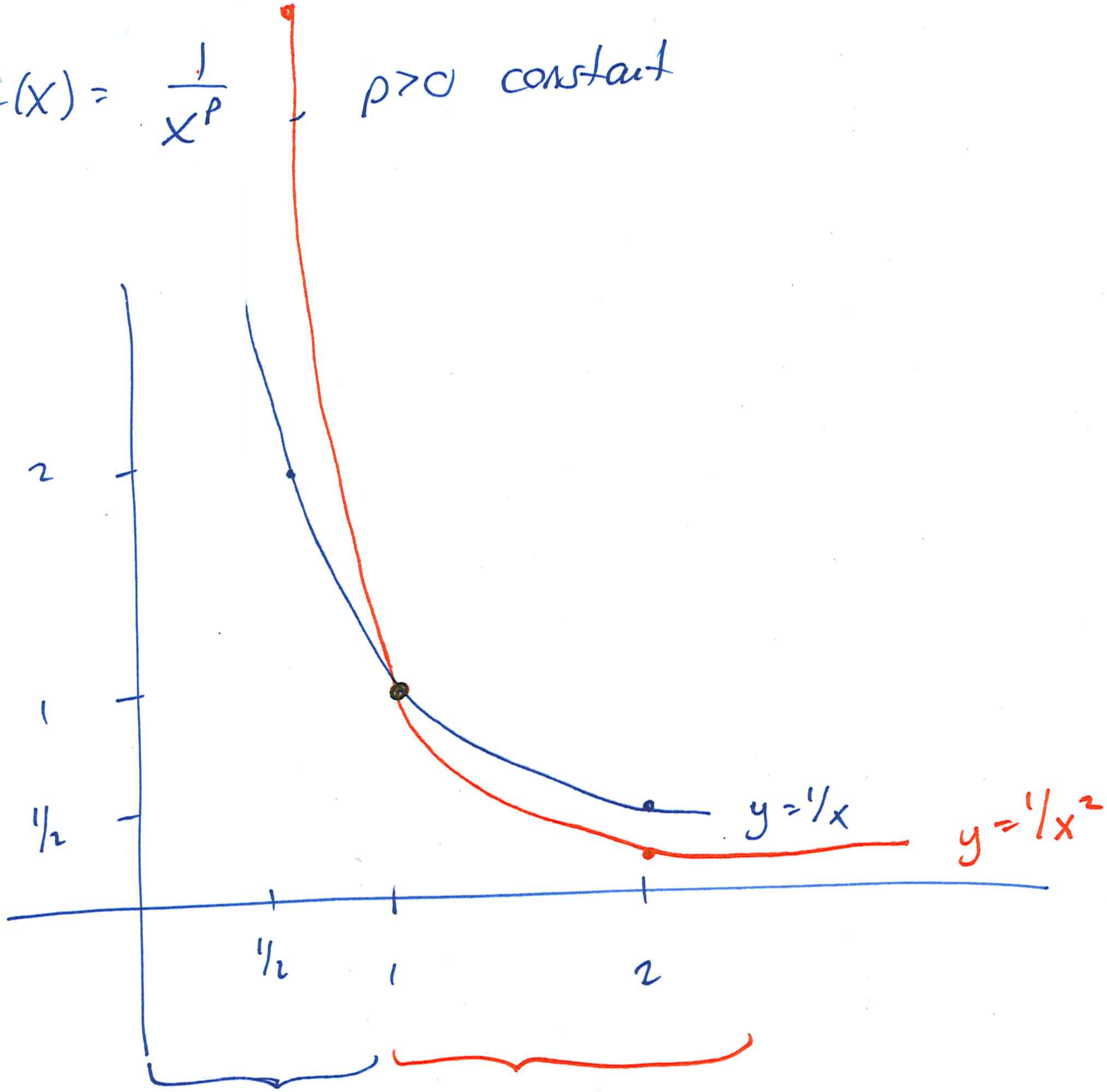
$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\left[\frac{-1}{x}\Big|_a^1\right] = \lim_{a\to 0^+}\left[\frac{-1}{1} - \underbrace{\frac{-1}{a}}_{\to\infty}\right] = \infty$$

DIVERGES

$f(x) = \dfrac{1}{x^p}$, $p > 0$ constant

$\dfrac{1}{(1/2)^2} = \dfrac{1}{1/4} = 4$

$\dfrac{1}{2^2} = \dfrac{1}{4}$



$\dfrac{1}{x} < \dfrac{1}{x^2}$ $\qquad$ $\dfrac{1}{x} > \dfrac{1}{x^2}$

ex $\displaystyle\int_0^1 \frac{1}{x^{0.999}}\,dx = \lim_{a\to 0^+}\left[\int_a^1 x^{-0.999}\,dx\right] =$

$$\lim_{a\to 0^+}\left[1000\,x^{0.001}\Big|_a^1\right] = \lim_{a\to 0^+}\left[1000-1000\cdot \underset{\to 0}{\cancel{a^{0.001}}}\right] = 1000$$

So: $\displaystyle\int_0^1 \frac{1}{x^{0.999}}\,dx$ converges

ex $\displaystyle\int_0^1 \frac{1}{x^{1.001}}\,dx = \lim_{a\to 0^+}\left[\int_a^1 x^{-1.001}\,dx\right] = \lim_{a\to 0^+}\left[-1000x^{-0.001}\Big|_a^1\right]$

$$= \lim_{a\to 0^+}\left[-1000+1000\,a^{-0.001}\right]$$

$$= \lim_{a\to 0^+}\left[-1000+\frac{1000}{a^{0.001}}\right]_{\to 0} = \infty$$

$0.001 = \frac{1}{1000}$

$\frac{1}{0.001} = \frac{1}{1/1000} = 1000$

So: $\displaystyle\int_0^1 \frac{1}{x^{1.001}}\,dx$ DIVERGES p-test:

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{converges} & \text{if } p < 1 \\ \text{diverges} & \text{if } p \geq 1 \end{cases}$$

p positive constant

$$\int_1^\infty \frac{1}{x^p}\,dx : \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$$

ex $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$: DIVERGES (by p-test)

$p = 1/2 < 1$

ex $\int_0^1 \frac{1}{x^{2.7}}\,dx$ DIVERGES

$p = 2.7 > 1$

(ex) $$\int_0^{17} \frac{1}{\sqrt{x}}\,dx$$

Conv or Div?

$$= \underbrace{\int_0^{1} \frac{1}{\sqrt{x}}\,dx}_{\text{determines conv/div}} + \underbrace{\int_1^{17} \frac{1}{\sqrt{x}}\,dx}_{\text{some number (not improper)}}$$

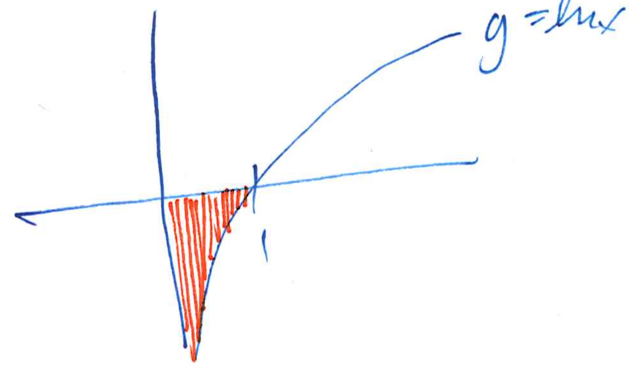
$p = 1/2 < 1$

converge

So: $\int_0^{17} \frac{1}{\sqrt{x}}\,dx$ converges.

LOL nevermind

ex $\displaystyle\int_0^1 \ln x \, dx$



Evaluate.

$$\int 1 \cdot \ln x \, dx = x\ln x - \int x \cdot \tfrac{1}{x}\, dx = x\ln x - \int 1 \, dx$$

$$= \boxed{x\ln x - x} + C$$

$u : \ln x \qquad du : \frac{1}{x}\, dx$

$dv : 1\, dx \qquad v : x$

$$\int_0^1 \ln x\, dx = \lim_{a\to 0^+}\left[\int_a^1 \ln x\, dx\right] = \lim_{a\to 0^+}\left[\left(1\overset{0}{\cancel{\ln 1}}-1\right) - (a\ln a - a)\right]$$

$$= \lim_{a\to 0^+}\left[-1 - a\ln a + a\right] = \lim_{a\to 0^+}\left[-1 - \underset{0}{a}\,(\underset{-\infty}{\ln a - 1})\right]$$

indeterminate form

$$= \lim_{a\to 0^+}\left[-1 - \underbrace{\frac{\ln a - 1}{1/a}}_{\frac{\infty}{\infty}}\right] = \lim_{a\to 0^+}\left[-1 + \frac{1/a}{+1/a^2}\right] = \lim_{a\to 0^+}\left[-1 + a\right] = \boxed{-1}$$

use L'Hospital

Ch 7.9 Differential Equations

(ex) $\boxed{y' = e^x}$ and $y(0) = 2$. What is y?

1st order differential equation

$y = \int e^x dx = e^x + C$, $\quad y = e^x + C$

$x=0$: $\quad 2 = e^0 + C$

$2 = 1 + C$

$C = 1$

So: $\boxed{y = e^x + 1}$

2nd order differential equation

(ex) $\boxed{y''(t) = 12t + 1}$, $y(0) = 1$, $y(1) = 10$. Find y

$y'(t) = \int 12t + 1\, dt = 6t^2 + t + C$

$y(t) = \int (6t^2 + t + C)\, dt = 2t^3 + \frac{1}{2}t^2 + Ct + D$

$y(t) = 2t^3 + \frac{1}{2}t^2 + Ct + D$

$1 = D \quad (y(0) = 1)$

$10 = 2 + \frac{1}{2} + C + D$

$10 = 2 + \frac{1}{2} + C + 1$

$C = 6.5$

$\boxed{y = 2t^3 + \frac{1}{2}t^2 + 6.5t + 1}$

(ex) $\boxed{y' = ky + b}$

where k, b constants
y function of ~~x~~ t

General Solution:

$y = Ce^{kt} - b/k$

for some C - constant

(ex) $y' = 3y + 7$, $y(2) = 5$

what is y?

$y = Ce^{3t} - 7/3$ for some C

[y(2)=5] $5 = C \cdot e^{3\cdot 2} - 7/3$ find C

$\frac{22}{3} = C \cdot e^{6}$

$\frac{22}{3 \cdot e^{6}} = C$

So: $y = \frac{22}{3e^{6}} \cdot e^{3t} - 7/3$

$$\boxed{y = \frac{22}{3} e^{3t-6} - 7/3}$$

Check: $y' = 3y + 7$

$\underline{y'} = \frac{22}{3} \cdot e^{3t-6} \cdot 3 = \underline{\underline{22e^{3t-6}}}$

$\underline{\underline{3y+7}} = 3(\underbrace{\frac{22}{3} e^{3t-6} - 7/3}_{y}) + 7$

$= 22e^{3t-6} - 7 + 7 = \underline{\underline{22e^{3t-6}}}$

TRUE: $y' = 3y + 7$ for this y

Check: $y(2)=5$

$$y(t)=\frac{22}{3}e^{3t-6}-7/3$$

$$y(2)=\frac{22}{3}e^{0}-7/3=\frac{22}{3}-\frac{7}{3}=\frac{15}{3}=5 \checkmark$$