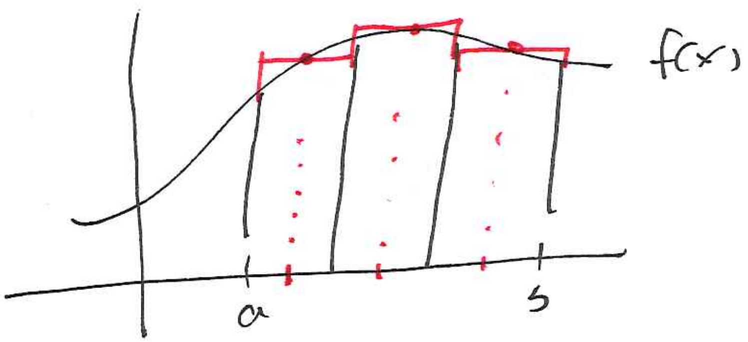
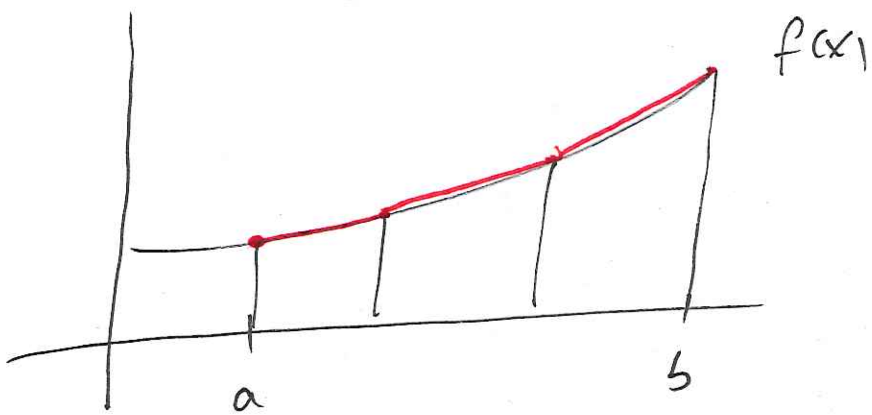
2 March 2017

Midpoint Rule (p. 559)



Approximating f(x)
by a constant
(in each interval)

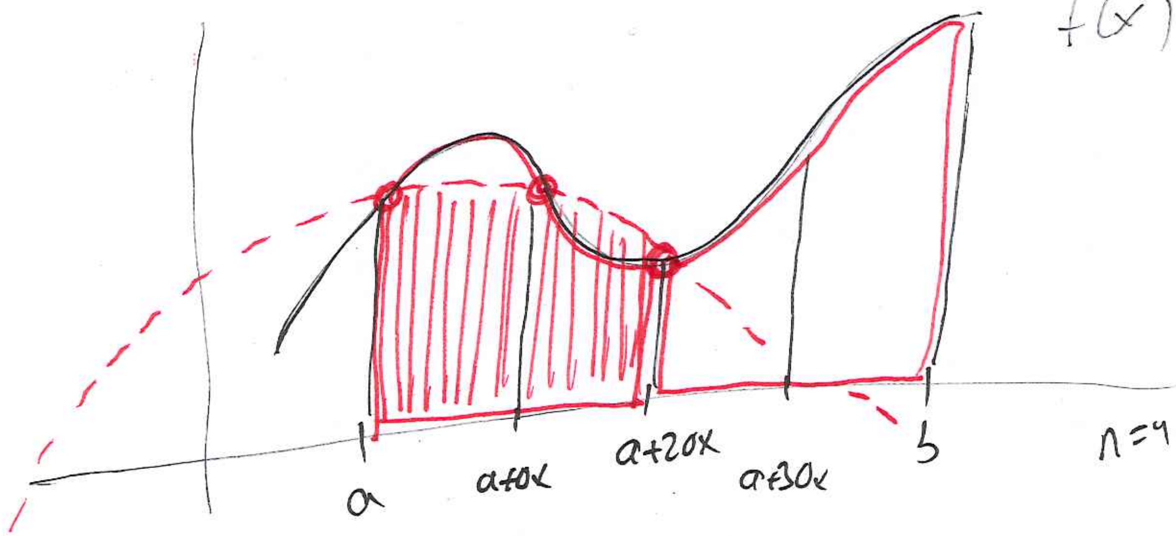
Trapezoid Rule (p. 560)



Approximating f(x)
by a line
(in each interval)

Simpson's Rule:

Approx f(x) by
a parabola
(in each interval)



Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 4f(x_{n-1}) + f(x_n) \Big]$$

$\Delta x = \frac{b-a}{n}$

$\boxed{\text{only when } n \text{ even}}$

(ex) Use Simpson's Rule, $n=4$ intervals

$$\int_0^8 \frac{1}{1+x^2}\,dx \approx \frac{2}{3}\left[1+4\cdot\frac{1}{5}+2\cdot\frac{1}{\cancel{10}17}+4\cdot\frac{1}{37}+\frac{1}{65}\right]$$

$f(0)=1$ $\quad f(2)=\frac{1}{5}$ $\quad f(4)=\frac{1}{\cancel{16}\,17}$ $\quad f(6)=\frac{1}{37}$ $\quad f(8)=\frac{1}{65}$

$\Delta x = \frac{b-a}{n} = \frac{8}{4} = 2$

$$0 \qquad \underbrace{2 \qquad 4}_{\Delta x = 2} \qquad 6 \qquad 8$$

(ex) Use Simpson's Rule, 8 intervals

$$\int_0^8 e^{x^2}\,dx \approx \frac{1}{3}\Big[e^0+4e^1+2e^4+4\cdot e^9+2e^{16}+4e^{25}+2e^{36}$$
$$+4e^{49}+e^{64}\Big]$$

$\Delta x = \frac{b-a}{n} = \frac{8-0}{8} = 1$

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8$$

Formulas for Error: formula sheet

Theorem 7.2, p565

(ex) Find error involved with approximating $\int_0^1 \sin(2x)\,dx$ using 10 intervals (all 3 methods)

$b = 1$ $\quad (b-a) = 1$

$a = 0$ $\quad \Delta x = \frac{b-a}{n} = \frac{1}{10}$

$n = 10$

$4 = k$

$16 = K$

$f(x) = \sin(2x)$

$f'(x) = 2\cos(2x)$

$\boxed{f''(x) = -4\sin(2x)}$ $\quad |-4\sin(2x)| \le 4$

$f'''(x) = -8\cos(2x)$

$\boxed{f^{(4)}(x) = 16\sin 2x}$ $\quad |16\sin(2x)| \le 16$

Using Midpoint: $|\text{Error}| \le \frac{k(b-a)}{24}(\Delta x)^2 = \frac{4\cdot 1}{24}\left(\frac{1}{10}\right)^2 = \frac{1}{6}\cdot\frac{1}{100} = \boxed{\frac{1}{600}}$

Using Trapezoid: $|\text{Error}| \le \frac{k(b-a)}{12}(\Delta x)^2 = \frac{4\cdot 1}{12}\left(\frac{1}{10}\right)^2 = \frac{1}{3}\cdot\frac{1}{100} = \boxed{\frac{1}{300}}$

Possible: (eg) error using MP: $\frac{1}{600}$

error using ⏢ : $\frac{1}{1000}$

Error)
Using Simpson's Rule: $\leq \dfrac{K(b-a)}{180}(\Delta x)^4$

$$= \frac{16(1)}{180}\left(\frac{1}{10}\right)^4 = \frac{1}{112{,}500}$$

ex) What is the error involved with Simpson's Rule approximating $\int_1^2 \frac{1}{x}\,dx$ using 6 intervals?

$$E_S \leq \frac{K(b-a)}{180}(\Delta x)^4 = \frac{24(2-1)}{180}\Big(\underbrace{\frac{2-1}{6}}_{\Delta x}\Big)^4$$

K : upper-bound on $|f^{(4)}(x)|$

$$= \frac{24}{180 \cdot 6^4} \stackrel{?}{=} \frac{1}{38{,}880}$$

$f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -x^{-2}$

$f''(x) = 2x^{-3}$

$f'''(x) = -6x^{-4}$

$f^{(4)}(x) = 24x^{-5} = \frac{24}{x^5}$

$\left|\frac{24}{x^5}\right| \leq \frac{24}{1} = 24$

Use $K = 24$

Note: $\int_1^2 \frac{1}{x}\,dx = \ln 2 - \ln 1 = \boxed{\ln 2}$ exact

Suppose you want to approx $\ln 2 = \int_1^2 \frac{1}{x}\,dx$

using midpoint rule, your error should be

at most 10^{-4}.

How many intervals do you need?

$$E_M \leq \frac{k(b-a)}{24}(\Delta x)^2 \underset{\text{WANT}}{\leq} 10^{-4}$$

$f(x) = \frac{1}{x}$

$f'(x) = \frac{-1}{x^2}$

$\boxed{f''(x) = \frac{2}{x^3}}$ → when x is between 1 & 2,

$|f''(x)| = \left|\frac{2}{x^3}\right| \leq \frac{2}{1} = 2$

Use: $\boxed{k = 2}$

$$\frac{k(b-a)}{24}(\Delta x)^2 \overset{want}{\leq} 10^{-4}$$

$$\frac{2(2-1)}{24}\left(\frac{\cancel{b-a}^{\,1}}{n}\right)^2 \leq \frac{1}{10^4}$$

$$\frac{1}{12}\cdot\frac{1}{n^2} \leq \frac{1}{10^4}$$

$$12n^2 \geq 10^4$$

$$n^2 \geq \frac{10^4}{12}$$

$$n \geq \sqrt{\frac{10^4}{12}} = \frac{100}{\sqrt{12}} \approx 28.8$$

Use $\boxed{29}$ intervals.

→ Suggested Probs §7.7

§7.8 Improper Integrals

Two ways for an integral to be improper:

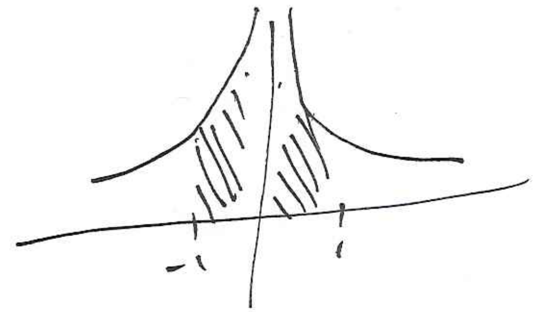
- infinite interval of integration
- integrand (f(x)) not bounded on region of integration

example: $\int_1^\infty \frac{1}{x^2}\,dx$

improper

$\int_0^1 \frac{1}{x^2}\,dx$, $\int_{-1}^1 \frac{1}{x^2}\,dx$

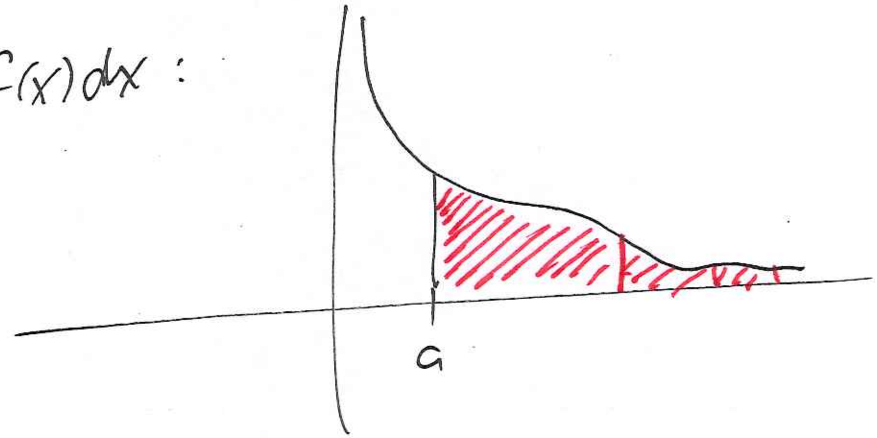
improper



Infinite Interval

We use a limit

$\int_a^{\infty} f(x)\,dx$:



ex $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$

$$= \lim_{b\to\infty}\left[\frac{-1}{x}\Big|_1^b \right]$$

$$= \lim_{b\to\infty}\left[\frac{-1}{b} - \frac{-1}{1}\right] = \lim_{b\to\infty}\left[\frac{-1}{b} + 1\right] = \boxed{1}$$

ex $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx \neq \lim_{a\to\infty} \int_{-a}^{a} \frac{1}{1+x^2}\,dx$

$$\|$$

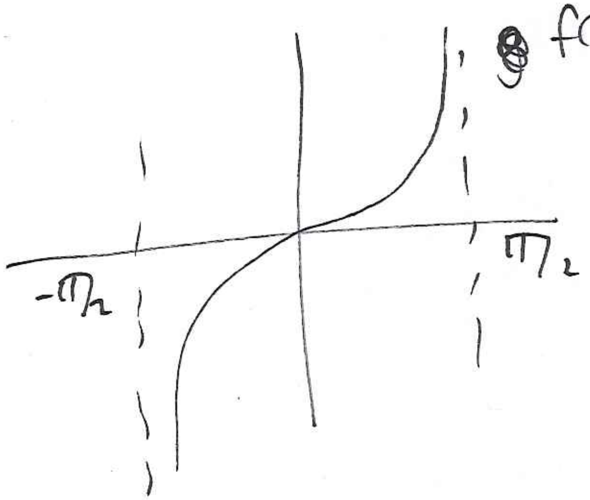
$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty}\left[\int_a^0 \frac{1}{1+x^2}\,dx\right] + \lim_{b\to\infty}\left[\int_0^b \frac{1}{1+x^2}\,dx\right]$$

$$= \lim_{a\to-\infty}\left[\overset{0}{\cancel{\arctan 0}} - \arctan a\right] + \lim_{b\to\infty}\left[\arctan b - \overset{0}{\cancel{\arctan 0}}\right]$$

$$= -\left[-\pi/2\right] + \pi/2 = \pi/2 + \pi/2 = \boxed{\pi}$$

Aside: $\arctan(x) = y$

means: $\tan(y) = x$



$$\left.\begin{array}{l}\text{As } y \to \pi/2, \\ \tan(y) \to \infty\end{array}\right\} \text{"}\arctan \infty\text{"} \text{"="} \pi/2$$

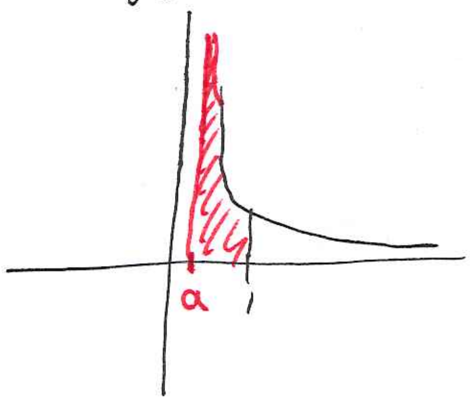
$$\lim_{x \to \infty} \arctan x = \pi/2$$

$$\lim_{x \to \pi/2^-} \tan x = \infty$$

$$\lim_{x \to -\infty} \arctan x = -\pi/2$$

Unbounded Function

ex) $\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\left[\int_a^1 \frac{1}{x^2}\,dx\right] = \lim_{a\to 0^+}\left[\frac{-1}{x}\Big|_a^1\right]$



$= \lim_{a\to 0^+}\left[\frac{-1}{1} - \frac{-1}{a}\right]$

$= \lim_{a\to 0^+}\left[-1 + \frac{1}{a}\right] = \infty$

We say this integral diverges

(limit doesn't exist!)

By the way: If limit gives a finite number, we say the integral converges.

(ex) $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\left[\int_a^1 x^{-1/2}\,dx\right]$

$= \lim_{a\to 0^+}\left[2x^{1/2}\Big|_a^1\right]$

$= \lim_{a\to 0^+}\left[2\sqrt{1} - 2\sqrt{a}\right]$

$= \boxed{2}$