March 1, 2017

Trig Substitutions: Completing the Square

$$\int \frac{1}{\sqrt{3-x^2+2x}}\,dx$$

$\leftarrow \sqrt{\text{quadratic}}$

no obvious sub,

int by parts

$$\boxed{\begin{aligned} 1-\sin^2\theta &= \cos^2\theta \\ 1+\tan^2\theta &= \sec^2\theta \\ \sec^2\theta - 1 &= \tan^2\theta \end{aligned}}$$

2 pieces → 1 piece

$3-x^2+2x$: 3 pieces

doesn't really look like any of our identities

$$-[\underbrace{x^2-2x}-3]$$

$$(x+a)^2 = \boxed{\begin{matrix} x^2+2ax \\ x^2-2x \end{matrix}} \begin{matrix} +a^2 \\ -3 \end{matrix}$$

$$a=-1$$

$$(x-1)^2 = \underbrace{x^2-2x}+1$$

$$= -[\underbrace{x^2-2x+1}_{\square}\ \underbrace{-1\ -3}_{\text{const}}]$$

$$= -[(x-1)^2-4]$$

$$= 4-(x-1)^2$$

$$\int \frac{1}{\sqrt{4-(x-1)^2}}\,dx$$

Quadratic:

$4-(x-1)^2$: 2 pieces

$1-\sin^2\theta = \cos^2\theta$

$4-4\sin^2\theta = 4\cos^2\theta$

Want a substitution that turns

$4-(x-1)^2$ into

$4-4\sin^2\theta$

$(x-1)^2 = 4\sin^2\theta$

$x-1 = 2\sin\theta$

$\boxed{x = 1+2\sin\theta}$ use this substitution

$dx = 2\cos\theta\,d\theta$

$$\int \frac{1}{\sqrt{4-(x-1)^2}}\,dx = \int \frac{1}{\sqrt{4-(2\sin\theta)^2}}\cdot 2\cos\theta\,d\theta = \int \frac{1}{\sqrt{4-4\sin^2\theta}}\cdot 2\cos\theta\,d\theta$$

$$= \int \frac{1}{\cancel{\sqrt{4\cos^2\theta}}}\,2\cos\theta\,d\theta = \int \frac{1}{2\cancel{\cos\theta}}\cdot 2\cancel{\cos}\theta\,d\theta = \int 1\,d\theta = \theta + C$$

$$= \boxed{\arcsin\left(\frac{x-1}{2}\right) + C}$$

$x-1 = 2\sin\theta$

$\frac{x-1}{2} = \sin\theta$

So $\theta = \arcsin\left(\frac{x-1}{2}\right)$

Ch 7.5: Partial Fractions

Motivation: $$\frac{1}{x+1} - \frac{1}{2x-1} = \frac{x-2}{(x+1)(2x-1)}$$

$$\int \frac{x-2}{(x+1)(2x-1)}\,dx = ???$$

$$\int \frac{1}{x+1} - \frac{1}{2x-1}\,dx = \int \frac{1}{x+1}\,dx - \int \frac{1}{2x-1}\,dx = \int \frac{1}{u}\,du - \int \frac{1}{2}\cdot\frac{1}{w}\,dw$$

etc

$u = x+1$
$du = dx$

$w = 2x-1$
$\frac{1}{2}dw = dx$

Method of Partial Fractions: Turn a rational function $\left(\frac{\text{polynomial}}{\text{polynomial}}\right)$ into a sum of fractions that are easy to integrate.

ALGEBRA

Case 1: Denominator is a power of a linear function

$$\frac{\text{numerator}}{(ax+b)^n} = \frac{A}{(ax+b)} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3} + \cdots + \frac{D}{(ax+b)^n}$$

num: degree < n
a, b constants
n whole #

ex $\int \frac{x^2}{(x-1)^3}\,dx$

$$\frac{x^2}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

what are A, B, C?

common denominator: $(x-1)^3$

$$\frac{x^2}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$

$$\begin{aligned} x^2 &= A(x-1)^2 + B(x-1) + C \\ &= A(x^2 - 2x + 1) + Bx - B + C \\ &= Ax^2 \underline{- 2Ax} + A + \underline{Bx} - B + C \\ &= Ax^2 + (-2A + B)x + (A - B + C) \end{aligned}$$

Match coefficients of powers of x

$$\underline{1}\cdot x^2 + \underline{0}\cdot x + 0 = \underline{A}x^2 + \underline{(-2A+B)}x + (A-B+C)$$

$\boxed{A=1}$

$-2A+B=0 \qquad -2+B=0 \qquad \boxed{B=2}$

$A-B+C=0$

$1-2+C=0$

$-1+C=0 \qquad \boxed{C=1}$

$$\int \frac{x^2}{(x-1)^3}\,dx = \int \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{(x-1)^3}\,dx \underset{\substack{u=x-1\\du=dx}}{=} \int \frac{1}{u} + \frac{2}{u^2} + \frac{1}{u^3}\,du \ldots$$

Rational $\left(\frac{\text{polynom}}{\text{polynom}}\right)$

ex $\int \frac{6x+7}{4x^2+20x+25}\,dx$

No obvious substitution

$$\frac{6x+7}{(2x+5)^2} = \frac{A}{(2x+5)} + \frac{B}{(2x+5)^2}$$

Find A & B)

common denom:
$(2x+5)^2$

$$= \frac{A(2x+5)+B}{(2x+5)^2}$$

$6x+7 = A(2x+5)+B$

$6x+7 = (2A)x + (5A+B)$

$7 = 5A+B \rightarrow 7 = 15+B$, $\boxed{B=-8}$

$6 = 2A \longrightarrow \boxed{A=3}$

$$\int \frac{3}{2x+5} + \frac{-8}{(2x+5)^2}\,dx \underset{u=2x+5}{=} \dots$$

Case 2: Distinct Linear Factors

Distinct → different

$$\frac{\text{numerator}}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)} = \frac{A}{a_1x+b_1} + \frac{B}{a_2x+b_2} + \cdots + \frac{C}{a_nx+b_n}$$

numerator: polynom, deg $< n$

a_i, b_i constant

all factors distinct

ex

$$\frac{7x+13}{2x^2+x-10} = \frac{7x+13}{(2x+5)(x-2)} = \frac{A}{2x+5} + \frac{B}{x-2}$$

find A & B

common denom $(2x+5)(x-2)$

$$7x+13 = A(x-2) + B(2x+5)$$

If $x=2$:

$14+13 = A(0) + B(9)$

$27 = 9B$

$\boxed{B=3}$

If $x=1$

$7+13 = A(-1) + 3(7)$

$20 = -A + 21$

$-1 = -A$

$\boxed{A=1}$

$$\frac{7x+13}{2x^2+x-10} = \underbrace{\frac{1}{2x+5} + \frac{3}{x-2}}_{\text{easier to integrate}}$$

Case 3: Mixture

ex $$\frac{2x^3-8x^2+12x-5}{x^4-2x^3+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$ Find A, B, C, D

Denominator: $x^2(x^2-2x+1) = x^2(x-1)^2$

common Denom $x^2(x-1)^2$

$$2x^3-8x^2+12x-5 = Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2$$

If $x=0$: $-5 = B(-1)^2$

$\boxed{B=-5}$

If $x=1$: $2-8+12-5 = D$

$\boxed{1=D}$

If $x=2$: $2\cdot 8-8\cdot 4+24-5 = 2A+B+4C+4D$

$-16+19 = 2A-5+4C+4$

$3 = 2A+4C-1$

$\boxed{4 = 2A+4C}$

If $x=-1$:

$-2-8-12-5 = A(-1)(-2)^2 + B(-2)^2 + C(-2)+D$

$-10-17 = -4A+4B-2C+D$

$-27 = -4A-20-2C+1$

$-27 = -4A-2C-19$

$\boxed{-8 = -4A-2C}$

$$4 = 2A + 4C \quad \} \quad 2 = A + 2C \rightarrow \boxed{A = 2 - 2C}$$

$$-8 = -4A - 2C$$

$$-8 = -4(2 - 2C) - 2C$$

solve

for C