Substitution Rule

(Chain Rule in reverse)

ex

$$f(x) = \sin(3x^2+x)$$

$$f'(x) = \cos(3x^2+x)\cdot(6x+1)$$

$$\int \cos(\overbrace{3x^2+x}^{\text{inside}})\cdot(\overbrace{6x+1}^{\text{deriv. of inside}})\,dx = \sin(3x^2+x) + C$$

ex

Chain rule:

$$\frac{d}{dx}\{f(g(x))\} = f'(g(x))\cdot g'(x)$$

Backwards:

$$\int f'(g(x))\cdot g'(x)\,dx = f(g(x)) + C$$

Mnemonic: change of variable

"dictionary"

$$\boxed{g(x) = u}$$

$$\frac{du}{dx} = g'(x)$$

$$\boxed{du = g'(x)\,dx}$$

$$\int f'(g(x))\cdot \underbrace{g'(x)\,dx}_{} =$$

$$\int f'(u)\cdot du = f(u) + C$$

$$= f(g(x)) + C$$

ex

$$\int e^{\boxed{\sin x}} \underbrace{\cos x\, dx}_{du} = \int e^u\, du = e^u + C = \boxed{e^{\sin x} + C}$$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\, dx$

Check: $\frac{d}{dx}\left\{ e^{\sin x} + C \right\} = e^{\sin x} \cdot \cos x$ ✓

ex

$$\int e^x \sin(e^x)\, dx = \int \sin u\, du = -\cos u + C = \boxed{-\cos(e^x) + C}$$

$u = e^x$

$\frac{du}{dx} = e^x$

$du = e^x\, dx$

(ex) $$\int \frac{e^x}{e^x+15}\,dx = \int \frac{1}{u}\cdot du = \ln|u|+C$$

$$= \ln|e^x+15|+C$$

$$= \boxed{\ln(e^x+15)+C}$$

$u = e^x+15$

$\frac{du}{dx} = e^x$

$du = e^x dx$

(ex) $$\int x \sec(x^2)\tan(x^2)\,dx = \int \frac{1}{2}\sec u\cdot\tan u\,du$$

$$= \frac{1}{2}\sec u + C = \boxed{\frac{1}{2}\sec(x^2)+C}$$

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

ex $\int \sin x \underbrace{\cos x \, dx}_{du} = \int u \cdot du = \frac{1}{2}u^2 + C$

$= \boxed{\frac{1}{2}(\sin x)^2 + C}$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x \, dx$

ex $\int x^4 \underbrace{(x^5+1)}^8 dx = \int \frac{1}{5} \cdot u^8 \, du = \frac{1}{5} \cdot \frac{1}{9} u^9 + C$

$u = x^5 + 1$

$\frac{du}{dx} = 5x^4$

$du = 5x^4 dx$

$\frac{1}{5} du = x^4 dx$

$= \frac{1}{45} u^9 + C$

$= \boxed{\frac{1}{45}(x^5+1)^9 + C}$

ex

$$\int \frac{s}{s-3}\,ds = \int \frac{u+3}{u}\,du = \int \frac{u}{u}+\frac{3}{u}\,du$$

$$= \int\left(1+\frac{3}{u}\right)du = u+3\ln|u|+C$$

$$= \boxed{s-3+3\ln|s-3|+C}$$

$u = s-3$

$du = ds$

$s = u+3$

ex

$$\int \frac{\sec^2(\sqrt{x}+1)}{\sqrt{x}}\,dx = \int 2\sec^2 u\,du = 2\tan u + C$$

$$= \boxed{2\tan(\sqrt{x}+1)+C}$$

$u = \sqrt{x}+1$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

ex $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int -\frac{1}{u} du$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$= -\ln|u| + C$

$= -\ln|\cos x| + C$

$= \ln|(\cos x)^{-1}| + C$

$= \ln\left|\frac{1}{\cos x}\right| + C$

$= \boxed{\ln|\sec x| + C}$

*memorize

ex $\int x^5\sqrt{x^3+1}\,dx = \int \underbrace{x^3}_{u-1}\underbrace{\sqrt{x^3+1}}_{\sqrt{u}}\underbrace{x^2dx}_{\frac{1}{3}du}$

$u = x^3+1 \to x^3 = u-1$

$\frac{du}{dx} = 3x^2$

$du = 3x^2dx$

$\frac{1}{3}du = x^2dx$

$= \int \frac{1}{3}(u-1)\sqrt{u}\,du$

$= \int \frac{1}{3}(u-1)u^{1/2}du$

$= \frac{1}{3}\int (u^{3/2} - u^{1/2})\,du$

$= \frac{1}{3}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C$

$= \frac{2}{15}u^{5/2} - \frac{2}{9}u^{3/2} + C$

$= \boxed{\frac{2}{15}(x^3+1)^{5/2} - \frac{2}{9}(x^3+1)^{3/2} + C}$

ex $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos x \, dx}{\sin^3 x} = \int_{1/\sqrt{2}}^{1} \frac{1}{u^3} \, du = \int_{1/\sqrt{2}}^{1} u^{-3} \, du =$

$u = \sin x$

$du = \cos x \, dx$

If $x = \pi/4$, $u = \sin(\pi/4) = 1/\sqrt{2}$

If $x = \pi/2$, $u = \sin(\pi/2) = 1$

$$\left.\frac{u^{-2}}{-2}\right|_{1/\sqrt{2}}^{1} =$$

$$\left(\frac{1^{-2}}{-2}\right) - \left(\frac{(1/\sqrt{2})^{-2}}{-2}\right)$$

$$= \frac{-1}{2} + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^{-2}$$

$$= \frac{-1}{2} + \frac{1}{2}(\sqrt{2})^2 = \frac{-1}{2} + \frac{1}{2}(2) = 1 - 1/2$$

$$= 1/2$$

ex $\displaystyle\int_0^2 \frac{2s}{s^2+1}\,ds = \int_1^5 \frac{1}{u}\,du = \ln|u|\Big|_1^5 = \ln 5 - \cancel{\ln 1}$

$= \boxed{\ln 5}$

$u = s^2+1$

$\frac{du}{ds} = 2s$

$du = 2s\,ds$

If $s=0$, $u = 0^2+1 = 1$

If $s=2$, $u = 2^2+1 = 5$

ex $\displaystyle\int_5^{10} \frac{8t+6}{2t^2+3t}\,dt = \int_{65}^{230} 2\,\frac{1}{u}\,du$

If $t=5$, $u = 2(5)^2+3\cdot 5 = 50+15 = 65$

If $t=10$, $u = 2\cdot 10^2+3\cdot 10 = 200+30 = 230$

$u = 2t^2+3t$

$\frac{du}{dt} = 4t+3$

$du = (4t+3)\,dt$

$2du = (8t+6)\,dt$

$= 2\ln|u|\Big|_{65}^{230} = 2\ln(230) - 2\ln(65)$

$= \boxed{2\ln\left(\frac{230}{65}\right)}$

(ex) $\int e^{x+e^x}\,dx = \int \underbrace{e^x e^{e^x}\,dx} = \int du = u + C = \boxed{e^{e^x} + C}$

$u = e^{(e^x)}$

$\frac{dy}{dx} = e^{(e^x)} \cdot e^x$

$du = e^{e^x} e^x\,dx$

Check:

$\frac{d}{dx}\left\{ e^{e^x} + c \right\} = e^{e^x} \cdot e^x$ ✓

(ex) $\int \frac{dx}{e^x + e^{-x}} \left(\frac{e^x}{e^x}\right) = \int \frac{e^x \, dx}{(e^x)^2 + 1} = \int \frac{1}{u^2+1}\,du = \arctan u + C$

du

$= \boxed{\arctan(e^x) + C}$

$u = e^x$

$du = e^x\,dx$

Ch 7.2: Integration By Parts

(product rule - backwards)

$$\frac{d}{dx}\left\{ u(x)\cdot v(x) \right\} = u'(x)\,v(x) + u(x)\cdot v'(x)$$

So: $$\int \left[u'(x)v(x) + u(x)v'(x)\right]dx = u(x)\cdot v(x) + C$$

$$\int u'(x)v(x)\,dx + \underbrace{\int u(x)v'(x)\,dx} = u(x)v(x) + C$$

So: $$\int u(x)\cdot v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx + C$$

Mnemonic: $$\boxed{\int u\,dv = uv - \int v\,du}$$ *memorize ex $\int x \sin x \, dx = -x\cos x - \int -\cos x (1)\, dx =$

$-x\cos x + \int \cos x \, dx$

$u: x$ $\quad du: 1\, dx$

$dv: \sin x dx$ $\quad \underbrace{v: -\cos x}$

$= \boxed{-x\cos x + \sin x + C}$

$\int u\,dv = uv - \int v\,du$

ex $\int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\, dx$

$u: \ln x$ $\quad du: \frac{1}{x}\, dx$

$dv: x\, dx$ $\quad v: \frac{1}{2}x^2$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x \, dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\left(\frac{1}{2}x^2\right) + C$

$= \boxed{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$