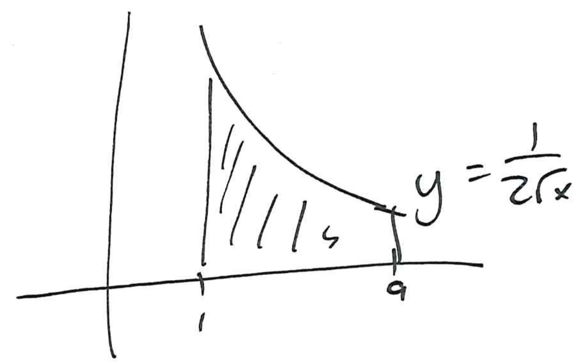
Last Time:

Fundamental Theorem of Calculus, Part II

If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

(ex) $$\int_1^9 \frac{1}{2\sqrt{x}}\,dx = \sqrt{9} - \sqrt{1} = 3-1 = \boxed{2}$$



Riemann sums; limit: hard
Geometry: hard
Let's use Fundamental Thm Part II

Recall: $F(x) = \sqrt{x} \Rightarrow F'(x) = \frac{1}{2\sqrt{x}}$

So $\sqrt{x}$ is an antideriv. of $\frac{1}{2\sqrt{x}}$

Antidifferentiation

Given a function f, what function F has $F' = f$

Notation: "indefinite integral"

$\int f(x)\,dx$: the most general antiderivative of $f(x)$ over an interval where its continuous

integral sign
no bounds ("indefinite")

$+C$

adding a constant doesn't change derivative

ex $\int \cos x\,dx = \sin x + C$

ex $\int \frac{1}{x}\,dx = \ln|x| + C$

ex $\int e^x\,dx = e^x + C$

ex $\int 15\cos x\,dx = 15\int \cos x\,dx$

$= 15\sin x + C$

Check: $\frac{d}{dx}\{15\sin x + C\}$

$= 15\cos x$ ✓

ex) $\int\left(\frac{1}{x}+\frac{1}{2\sqrt{x}}-e^x\right)dx = \boxed{\ln|x|+\sqrt{x}-e^x+C}$

Check: $\frac{d}{dx}\left\{\ln|x|+\sqrt{x}-e^x+C\right\} = \boxed{\frac{1}{x}+\frac{1}{2\sqrt{x}}-e^x}$ ✓

$f(x)$	$f'(x)$
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^n (n constant)	nx^{n-1}

$f(x)$	$\int f(x)dx$
1	$x+C$
x	$\frac{1}{2}x^2+C$
x^2	$\frac{1}{3}x^3+C$
x^3	$\frac{1}{4}x^4+C$
x^n	$\underbrace{\frac{1}{n+1}x^{n+1}+C}_{\text{if } n\neq -1}$
$x^{-1}=\frac{1}{x}$	$\ln\|x\|+C$

Check: $\frac{d}{dx}\left\{\frac{1}{2}x^2\right\} = \frac{1}{2}(2)x = x$

Check: $\frac{d}{dx}\left\{\frac{1}{n+1}\cdot x^{n+1}+C\right\}$

$= \frac{1}{\cancel{n+1}}\cdot\cancel{(n+1)}x^n = x^n$

When using FTC : can use <u>any</u> antideriv

ex $$\int_1^9 \frac{1}{2\sqrt{x}}\,dx = (\sqrt{9}+17) - (\sqrt{1}+17) = 3\cancel{+17}-1\cancel{-17} = 2$$

$F(x) = \sqrt{x}+17$ (note: $F'(x) = \frac{1}{2\sqrt{x}}$)

ex $$\int x^{-0.999}\,dx = \frac{x^{0.001}}{0.001} + C$$

$$\int x^{-1.001}\,dx = \frac{x^{-0.001}}{-0.001} + C$$

$$\int x^{-1}\,dx = \ln|x| + C$$

(ex) $\int \sqrt[3]{x}\,dx = \int x^{1/3}\,dx = \frac{3}{4}x^{4/3} + C$

(ex) $\int (5x^2 - e^x + 2)\,dx = \int 5x^2\,dx - \int e^x\,dx + \int 2\,dx$

$= 5\int x^2\,dx - \int e^x + \int 2\,dx = 5\cdot\frac{x^3}{3} - e^x + 2x + C$

ex $f(x) = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

(a constant)

Recall: $\frac{d}{dx}\{\arctan x\} = \frac{1}{1+x^2}$

Using Chain Rule:

$$f'(x) = \frac{d}{dx}\left\{\frac{1}{a}\cdot\arctan\left(\frac{x}{a}\right) + c\right\}$$

$$= \frac{1}{a}\cdot\left(\frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a}\right)$$

$$= \frac{1}{a^2}\cdot\left(\frac{1}{1+\frac{x^2}{a^2}}\right) = \frac{1}{a^2+x^2}$$

$\frac{1}{2} = \frac{1}{1+1} \neq \frac{1}{1} + \frac{1}{1} = 2$

So: $\boxed{\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C}$

Compare: $\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = -x^{-1} + C$

ex Note: $\frac{d}{dx}\{\arcsin x\} = \frac{1}{\sqrt{1-x^2}}$

(let $a>0$)

Calculate + simplify: $\frac{d}{dx}\{\arcsin(\frac{x}{a}) + c\} =$

$$\frac{1}{\sqrt{1-(\frac{x}{a})^2}} \cdot \frac{1}{a} = \frac{1}{a\sqrt{1-(\frac{x}{a})^2}} = \frac{1}{\sqrt{a^2}\sqrt{1-(\frac{x}{a})^2}}$$

$$= \frac{1}{\sqrt{a^2-a^2(\frac{x}{a})^2}} = \frac{1}{\sqrt{a^2-x^2}}$$

So: $\boxed{\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin(\frac{x}{a}) + C}$

Tidbit: An odd function behaves like this:

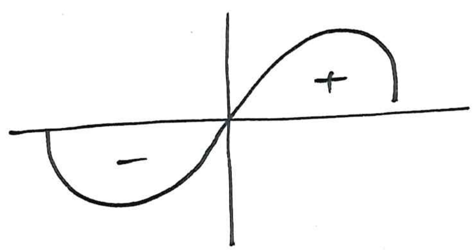
$$f(-x) = -f(x)$$

ex $f(x) = x^3$: $f(-2) = (-2)^3 = -8 = -(2^3) = -f(2)$

$g(x) = \sin x$: $\sin(-x) = -\sin(x)$

eg $\sin(-\pi/2) = -1 = -\sin(\pi/2)$

$\int_{-\pi}^{\pi} \sin x \, dx = 0$ b/c $\sin x$ is odd



In general, if $f(x)$ is odd: $\int_{-a}^{a} f(x)dx = 0$ for any real a,

$f(x)$ continuous on $[-a, a]$

Substitution Rule

chain rule backwards

chain rule

ex $\frac{d}{dx}\{\sin(3x^2)\} = \cos(3x^2)\cdot 6x$

$\frac{d}{dx}\{f(g(x))\} = f'(g(x))\cdot g'(x)$

So: $\int \cos(3x^2)\cdot 6x\,dx = \sin(3x^2)+C$

So: $\int f'(g(x))\cdot g'(x)\,dx = f(g(x))+C$

Mnemonic: $\boxed{u = g(x)}$

$\frac{du}{dx} = g'(x)$

$\boxed{du = g'(x)\,dx}$

use as dictionary to translate

$\int f'(\underbrace{g(x)}_{})\cdot \underbrace{g'(x)\,dx}_{} = \int f'(u)\,du = f(u)+C$

$= f(g(x))+C$

ex $\int e^{\sin x} \underbrace{\cos x \, dx} = \int e^{u} \, du = e^{u} + C = \boxed{e^{\sin x} + C}$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x \, dx$

Check: $\frac{d}{dx} \{ e^{\sin x} + c \} =$

$e^{\sin x} \cdot \cos x$