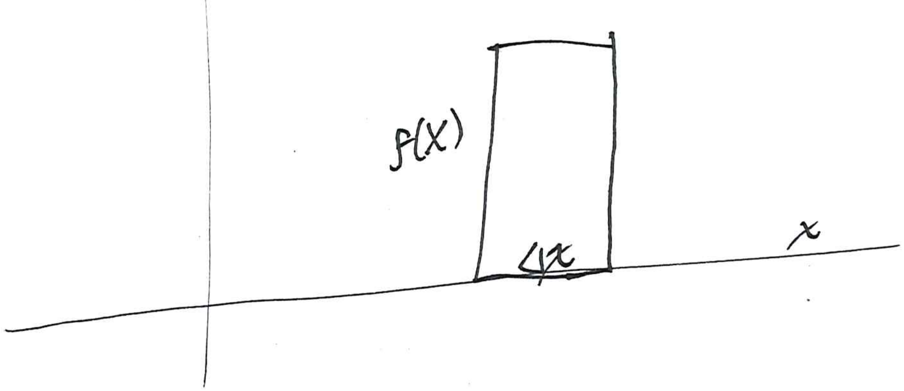
Last time, we calculated the area under the curve $y = f(x)$ over $[a,b]$ as:

$$\lim_{n \to \infty} \sum_{k=1}^{n} \Delta x \cdot f(x_k^*),$$

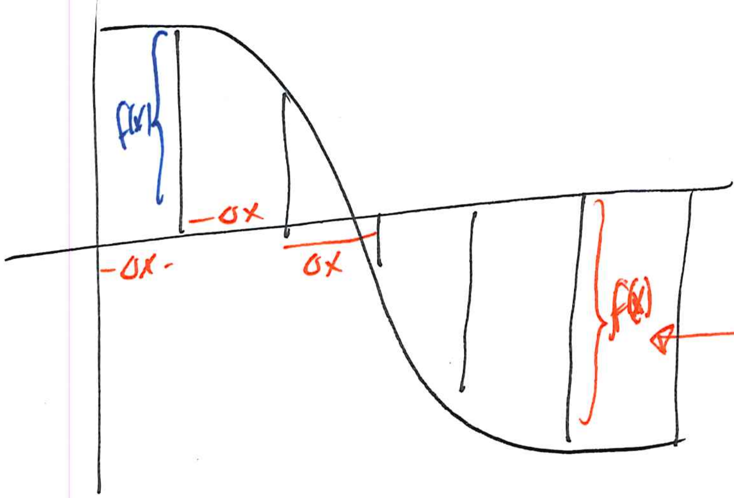
where $\Delta x = \frac{b-a}{n}$ and x_k^* was between $a+(k-1)\Delta x$ and $a+k\Delta x$

because, for each rectangle:



Q: What if $f(x) < 0$?

Actually calculating "signed area" between $f(x)$ + x-axis

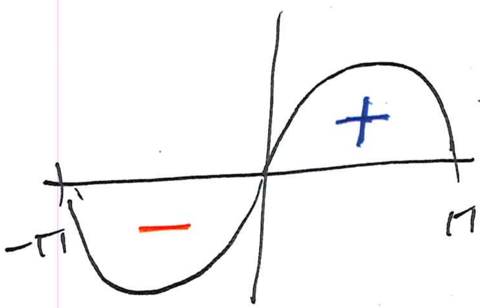


Area: (width)(height)

$\Delta x \cdot (-f(x))$

$\Delta x \cdot f(x) = -$ Area

ex) $\lim_{n \to \infty} \sum_{k=1}^{n} \Delta x \, f(x_k^*) \boxed{= 0}$, when $f(x) = \sin x$, $[-\pi, \pi]$



pos & neg areas will cancel out

Notation: Definite Integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x \cdot f(x_k^*)$$

where $\Delta x = \frac{b-a}{n}$

etc

$\int$: integral sign
"s" like "Σ"
for "sum"

a, b : "bounds"
bounds make an integral "definite"

dx : comes from
$\lim_{n\to\infty} \Delta x$
"differential"
tiny piece of
x-axis

Properties

1. $\int_a^b c \cdot f(x)\,dx = c \int_a^b f(x)\,dx$

c constant (just like with Σ)

2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

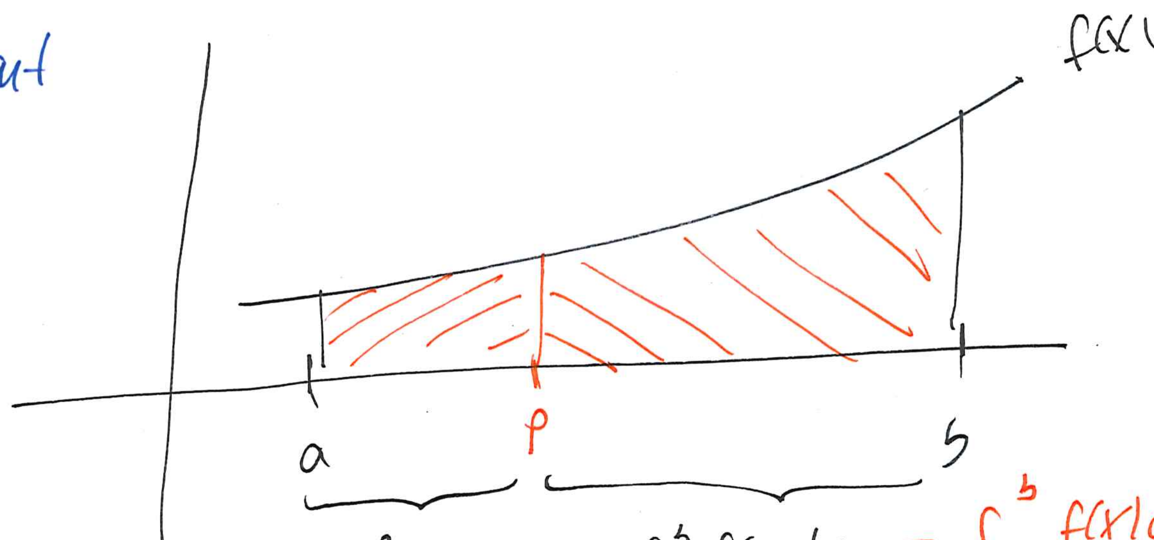
(like with Σ)

3. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Why? $\Delta x = \frac{b-a}{n} = -\left(\frac{a-b}{n}\right)$

4. $\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$

Let p - constant

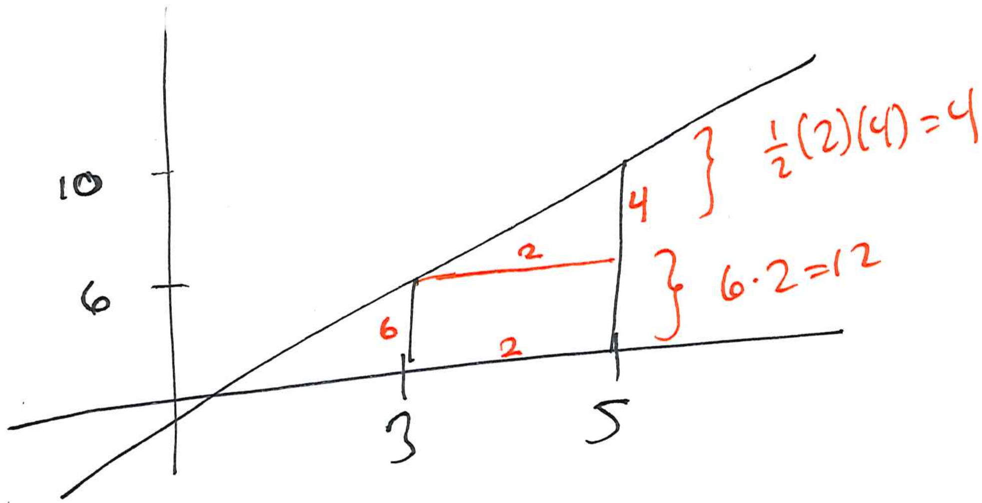


$\int_a^p f(x)\,dx + \int_p^b f(x)\,dx = \int_a^b f(x)\,dx$

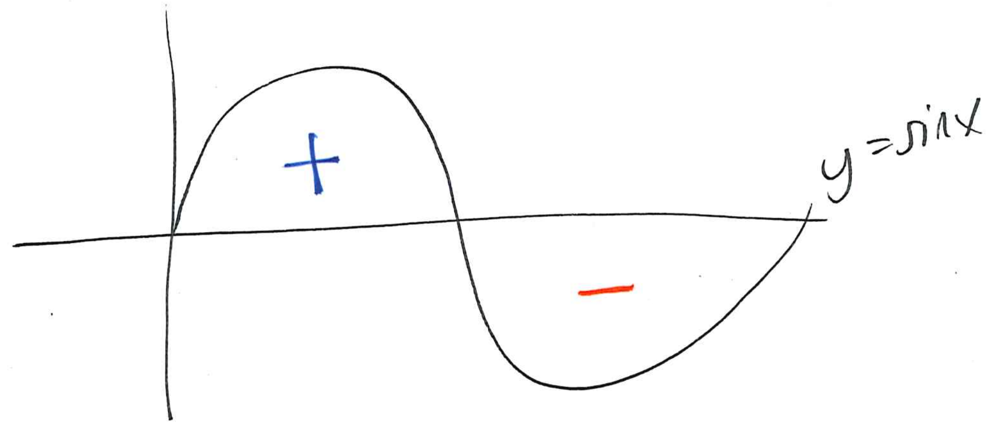
Use geometry to evaluate definite integrals

(ex) $\int_3^5 2x\,dx =$

$12+4=16$



(ex) $\int_0^{2\pi} \sin x\,dx = 0$



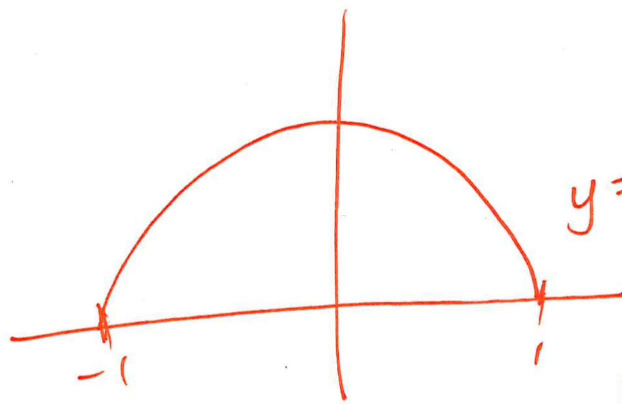
ex $\int_{-1}^{1} \sqrt{1-x^2}\,dx = \pi/2$

$y = \sqrt{1-x^2}$ $\quad\Big\|\quad$ $y \geq 0$ +

$-1 \leq x \leq 1$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1$

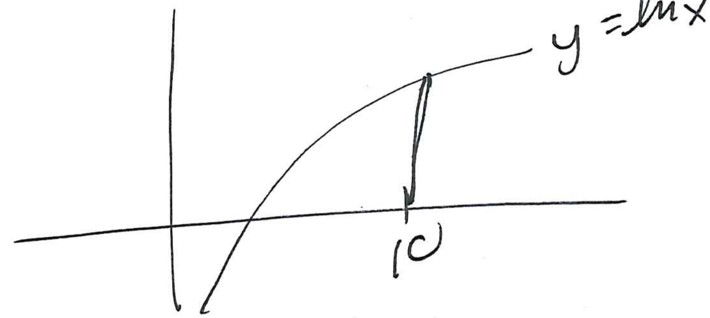


$y = \sqrt{1-x^2}$

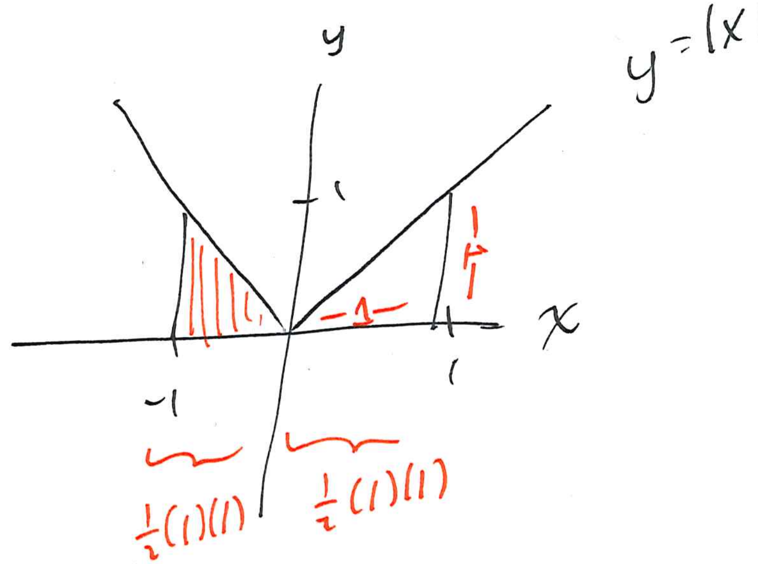
semicircle

Area: $\frac{1}{2}\pi r^2 = \pi/2$

ex $\int_{10}^{10} \ln x \, dx = 0$



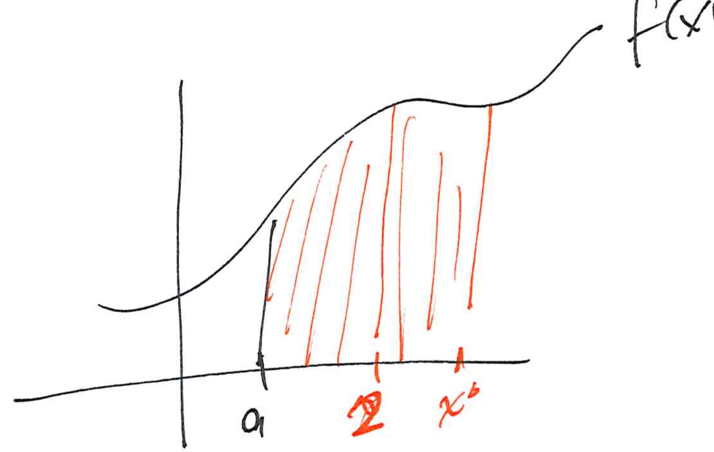
ex $\int_{-1}^{1} |x| \, dx = 1$



Chapter 5.3: Fundamental Theorem of Calculus

Area function:

$$A(x) = \int_a^x f(t)\,dt$$



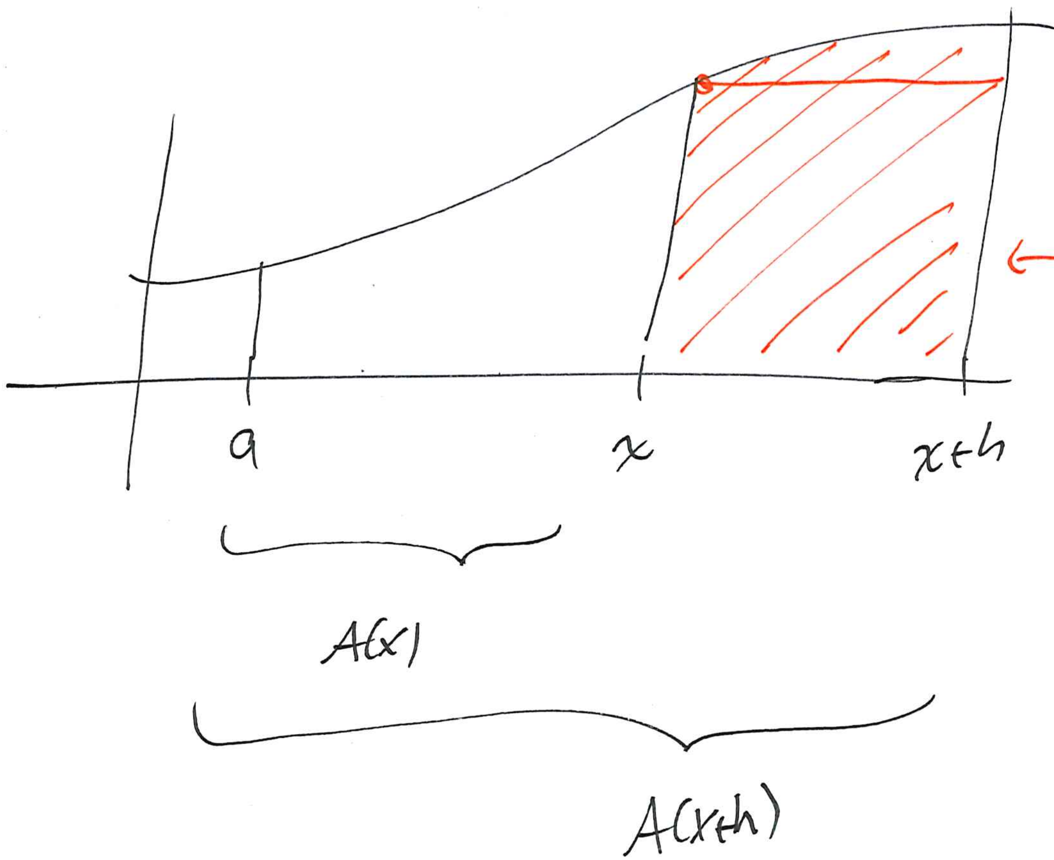
$A(2) = \int_a^2 f(t)\,dt$

$A(2)$: area from a to 2

$A(3)$: area from a to 3

etc

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h\to 0} \frac{h\,f(x)}{h} = f(x)$$



$A(x+h) - A(x) \approx (h)\, f(x)$

base

about its height

As $h \to 0$,

$$A(x+h) - A(x) \xrightarrow[h\to 0]{} h\,f(x)$$

Fundamental Theorem of Calculus, Part I

If f is continuous on $[a,b]$, then the area function

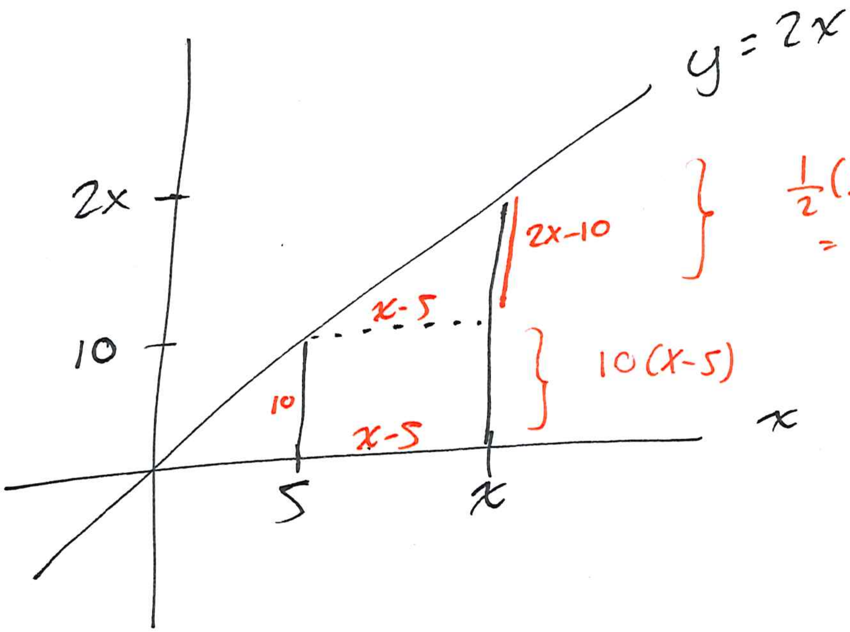
$$A(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$ & differentiable on (a,b)

and:

$$A'(x) = f(x)$$

(a) $A(x) = \int_5^x 2t \, dt = x^2 - 25$



$\frac{1}{2}(x-5)(2x-10)$
$= (x-5)^2$

Area: $10(x-5) + (x-5)^2$

$= (x-5)(10 + x - 5)$

$= (x+5)(x-5)$

$= x^2 - 25$

$A'(x) = \frac{d}{dx}\{x^2 - 25\} = 2x$

$$A(x) = \int_a^x f(t)\,dt$$

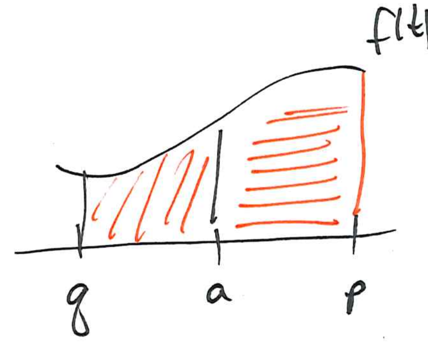
Fundamental Theorem of Calculus, Part I

FTC (I): I know $A'(x) = f(x)$

Let $F(x)$ be some function with $F'(x) = f(x)$

Since F & A have same derivative,

$F(x) = A(x) + C$ for some constant C



Calculate:

$$F(p) - F(q) = [A(p) + \cancel{C}] - [A(q) + \cancel{C}]$$

$$= A(p) - A(q) = \int_a^p f(t)\,dt - \int_a^q f(t)\,dt$$

$$= \int_a^p f(t)\,dt + \int_q^a f(t)\,dt = \int_q^a f(t)\,dt + \int_a^p f(t)\,dt$$

$$= \boxed{\int_q^p f(t)\,dt}$$

Fundamental Theorem of Calculus (II)

If f is continuous on $[a,b]$, and

F is <u>any</u> antiderivative of f on $[a,b]$

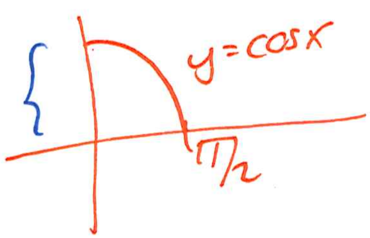
(I mean: $F'(x) = f(x)$)

then:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

ex $\int_0^{\pi/2} \cos x\,dx = \sin(\pi/2) - \sin(0) = 1 - 0 = 1$

$A = 1$



$f(x) = \cos x$

$F(x) = \sin x$