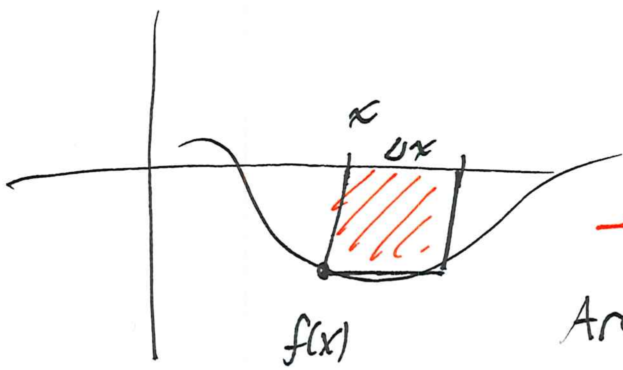
Last time, we said:

Area under curve $y = f(x)$, $[a, b]$ is

$$\lim_{n \to \infty} \sum_{k=1}^{n} \Delta x \cdot f(x_k^*) , \qquad \Delta x = \frac{b-a}{n}$$

x_k^* is between $a + (k-1)\Delta x$ + $a + k\Delta x$

Q: What if $f(x) < 0$?



– Area

Area: (width)(height)

$= \Delta x \, |f(x)|$

$= \Delta x \, (-f(x))$

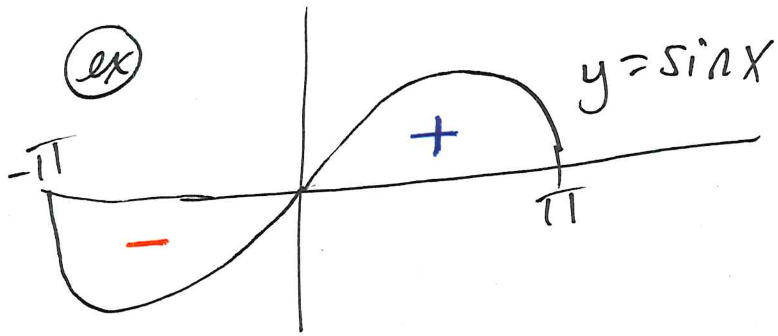
$= -\Delta x \, f(x)$

Actually calculating: "net area"

Area above axis

– area below axis

ex



$f(x) = \sin x$

$[-\pi, \pi]$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Delta x \cdot f(x_k^*) = 0$$

(positive area exactly cancels out negative area)

Notation: Definite Integrals

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x \cdot f(x_k^*), \quad \text{where } \Delta x = \frac{b-a}{n}$$

etc.

dx : "differential"
$\lim_{n\to\infty} \Delta x$

a, b : bounds
w/o bounds, integral is "indefinite"

$\int$ "integral sign"
elongated s for "sum"
same as Σ

Properties of Definit Integral

1. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

   why? $\Delta x = \frac{b-a}{n} = -\left[\frac{a-b}{n}\right]$

2. $$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

   why? $\sum_{k=1}^{n}(f+g) = \sum_{k=1}^{n} f + \sum_{k=1}^{n} g$

3. $$\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$$

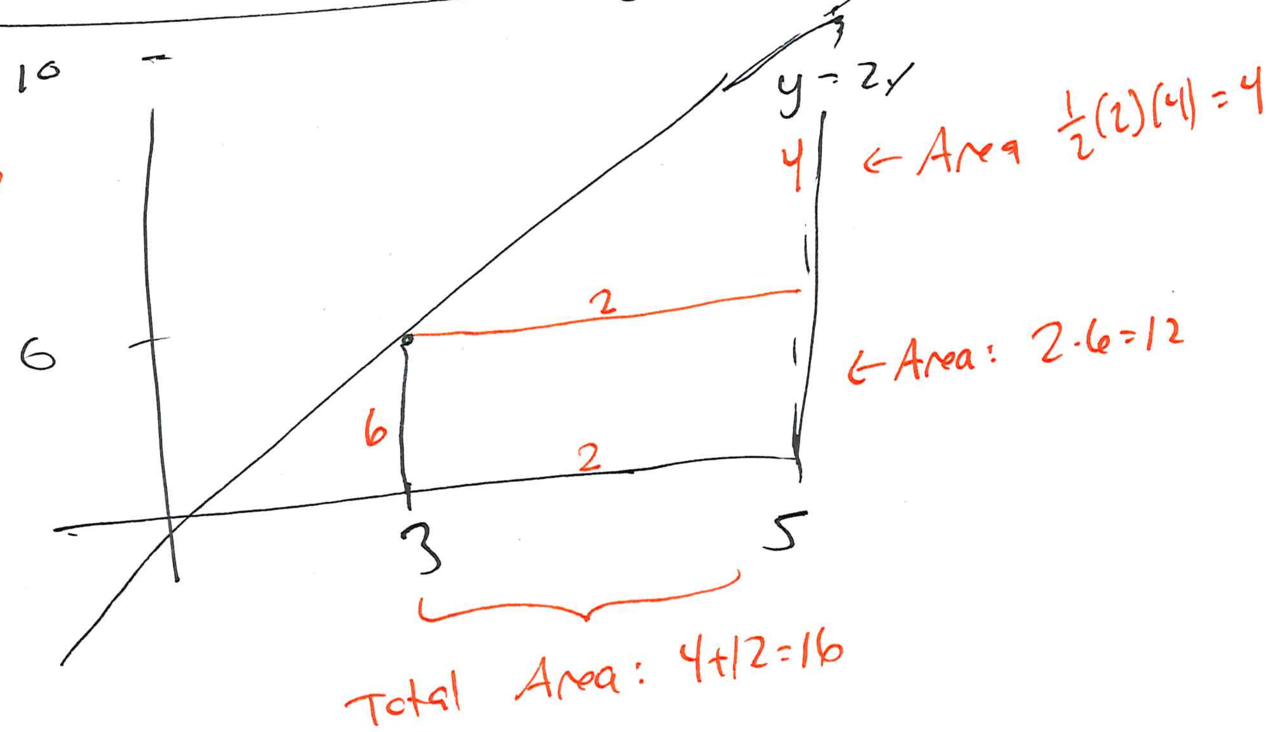
   c-constant

4. $$\int_a^b f(x)\,dx = \int_a^m f(x)\,dx + \int_m^b f(x)\,dx$$

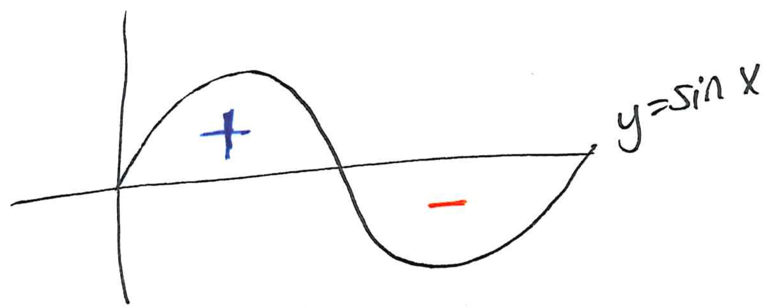
why?

Evaluating Definite Integrals Using Geometry

ex) $\int_3^5 2x\,dx = 16$



ex) $\int_0^{2\pi} \sin x\,dx = 0$

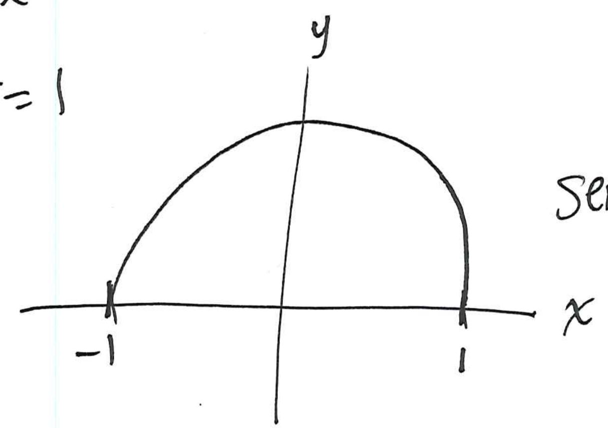


(ex) $\int_{-1}^{1} \sqrt{1-x^2}\,dx = \pi/2$

$y = \sqrt{1-x^2}$ $\quad y \geq 0$

$y^2 = 1-x^2$

$x^2+y^2 = 1$

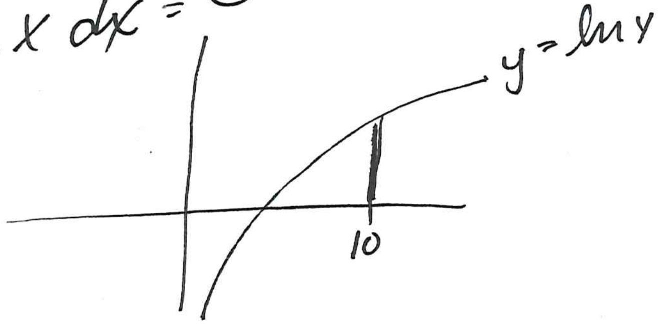


semicircle

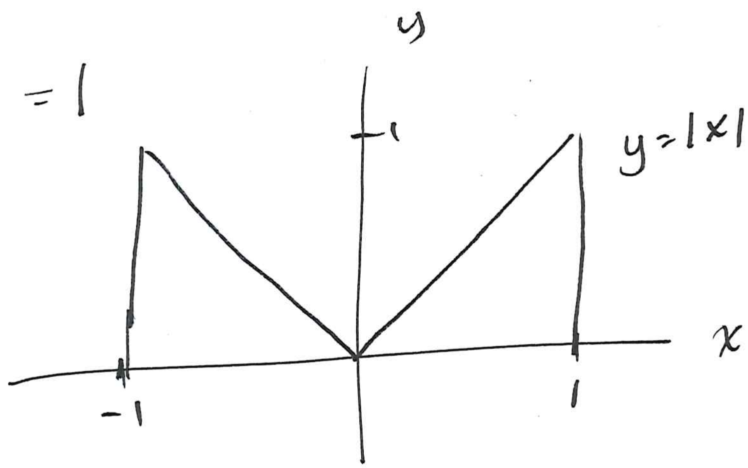
$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(1) = \pi/2$

(ex) $\int_{10}^{10} \ln x\,dx = 0$

no width!



(ex) $\int_{-1}^{1} |x|\,dx = 1$

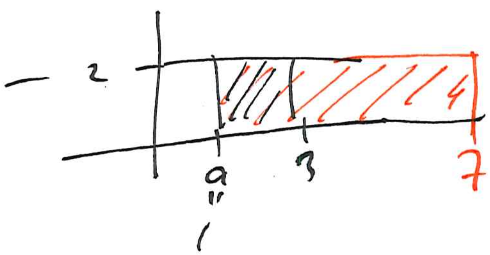


Ch 5.3 Fundamental Theorem of Calculus

Area function: $A(x) = \int_a^x f(t)\,dt$

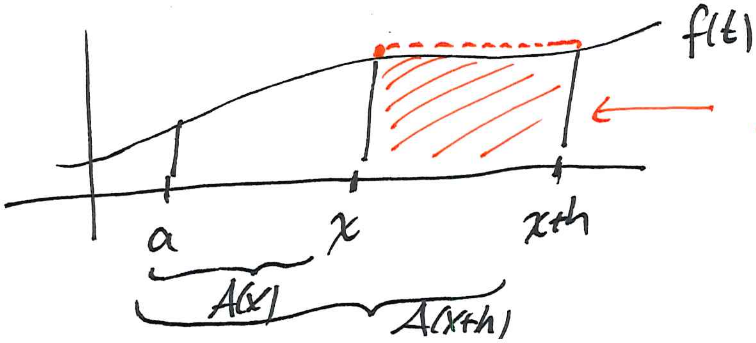
(ex) If $f(t) = 2$ $a = 1$

$A(3) = 4$ $A(7) = 12$



Derivative of Area Function:

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} \frac{h \cdot f(x)}{h} = f(x)$$



$A(x+h) - A(x) \approx h \cdot f(x)$

As $h \to 0$, $A(x+h) - A(x) \longrightarrow h \cdot f(x)$

Fundamental Theorem of Calculus, Part 1:

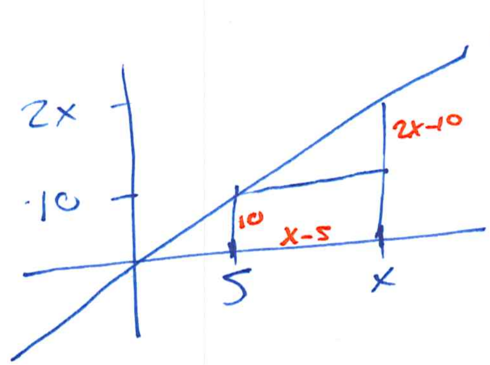
If f is continuous on $[a,b]$, then the area function

$A(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$

and differentiable on (a,b), and

$$A'(x) = f(x)$$

ex $A(x) = \int_5^x 2t\,dt$



} $\frac{1}{2}(x-5)(2x-10)$

} $(x-5)10$

$= 10(x-5) + \frac{1}{2}(x-5)(2x-10)$

$= 10(x-5) + (x-5)(x-5)$

$= (x-5)(10+x-5)$

$= (x-5)(x+5)$

$= x^2-25$

NOTICE: $A(x) = x^2-25$

$A'(x) = 2x = f(x)$

We are considering $A(x) = \int_p^x f(t)\,dt$

FTC(I): $A'(x) = f(x)$

Lots of functions have $f(x)$ as derivative.

Take any function $F(x)$ such that $F'(x) = f(x)$.

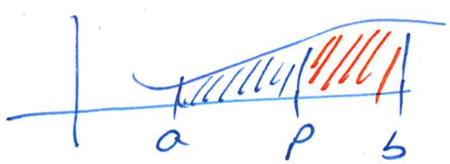
Since A, F have same derivative, they only differ by some constant, say C:

$F(x) = A(x) + C$

Notice: $F(b) - F(a) = [A(b) + \cancel{C}] - [A(a) + \cancel{C}] = A(b) - A(a)$

$= \int_p^b f(t)\,dt - \int_p^a f(t)\,dt = \int_p^b f(t)\,dt + \int_a^p f(t)\,dt$

$= \int_a^p f(t)\,dt + \int_p^b f(t)\,dt = \int_a^b f(t)\,dt$



Fundamental Theorem of Calculus, Part II

If f is continuous on $[a,b]$, and F is any antiderivative of f (I mean: $F'(x) = f(x)$), then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

ex $\displaystyle\int_5^{10} 2x\,dx = 10^2 - 5^2 = \boxed{75}$

could use geometry
could also use FTC

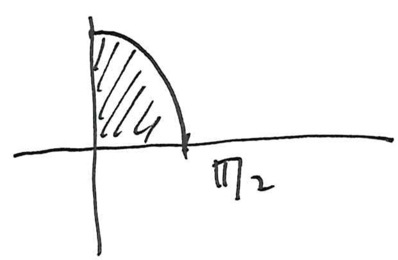
$f(x) = 2x$

$F(x) = x^2$

ex $\displaystyle\int_0^{\pi/2} \cos x\,dx = \sin(\pi/2) - \sin(0)$
$= 1 - 0 = \boxed{1}$

$f(x) = \cos x$

$F(x) = \sin x$



$$\int \cos x \, dx = \sin x + C$$

$$\int 15\cos x \, dx = 15\sin x + C$$

Constant Powers of x:

$f(x)$	$f'(x)$
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^n	nx^{n-1}

$f(x)$	$\int f(x)\,dx$
1	$x + C$
x	$\frac{1}{2}x^2 + C$
x^2	$\frac{1}{3}x^3 + C$
x^3	$\frac{1}{4}x^4 + C$
x^n	$\frac{1}{n+1}x^{n+1} + C$ if $n \neq -1$
$x^{-1} = \frac{1}{x}$	$\ln\lvert x\rvert + C$

$$\int \sqrt{x}\, dx = \int x^{1/2} dx = \frac{2}{3} x^{3/2} + C$$

indefinite (no bounds)

$$\int_1^9 \sqrt{x}\, dx = \frac{2}{3}(9)^{3/2} - \frac{2}{3}(1)^{3/2} = \frac{2}{3} \cdot \sqrt{9}^3 - \frac{2}{3} = \frac{2}{3} \cdot 27 - 2/3$$

$$= \frac{2}{3}(26)$$

definite integral -

area

ex $\frac{d}{dx}\{\arctan x\} = \frac{1}{1+x^2}$, so: $\int \frac{1}{1+x^2}\,dx = \arctan x + C$

Note: This is a little surprising:

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = -x^{-1} + C = \frac{-1}{x} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\frac{d}{dx}\left\{\frac{1}{a}\arctan\left(\frac{x}{a}\right)\right\} = \frac{1}{a}\cdot\frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a} = \frac{1}{a^2\left(1+\frac{x^2}{a^2}\right)} = \frac{1}{a^2+x^2}$$

(a constant)

So: $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

ex) $\frac{d}{dx}\{\arcsin x\} = \frac{1}{\sqrt{1-x^2}}$

$$\frac{d}{dx}\left\{\arcsin\left(\frac{x}{a}\right)\right\} = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{1}{a} = \frac{1}{\sqrt{a^2}\cdot\sqrt{1-\frac{x^2}{a^2}}}$$

$(a>0)$

$$= \frac{1}{\sqrt{a^2 - \cancel{a^2}\cdot\frac{x^2}{\cancel{a^2}}}} = \frac{1}{\sqrt{a^2-x^2}}$$

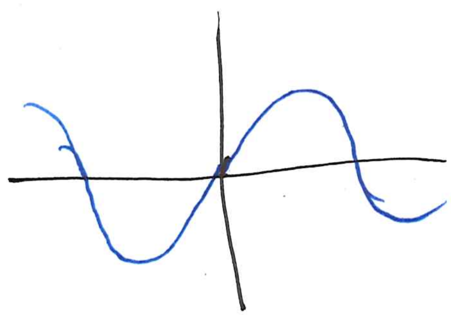
So: $\boxed{\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C}$

Even + Odd Functions

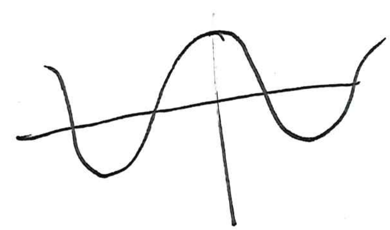
Odd function: $f(-x) = -f(x)$

ex: $f(x) = \sin x$

$f(x) = x^3 \longrightarrow f(-2) = -8 = -f(2)$



Odd Function: $\int_{-a}^{a} f(x)\,dx = 0$



Even function: $f(-x) = f(x)$

ex: $\cos x$

$f(x) = x^2$ $\qquad f(-2) = 4 = f(2)$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$