Feb 28

Ch 7.5 : Partial Fractions

Motivation: Fact: $\dfrac{1}{x+1} - \dfrac{1}{2x-1} = \dfrac{x-2}{(x+1)(2x-1)}$

$\displaystyle\int \frac{1}{x+1} - \frac{1}{2x-1}\,dx$: easy enough

$$\underbrace{\int \frac{1}{x+1}\,dx}_{u=x+1} - \underbrace{\int \frac{1}{2x-1}\,dx}_{u=2x-1} \quad \text{etc}$$

$\displaystyle\int \frac{x-2}{(x+1)(2x-1)}\,dx$: pretty tough

Method of Partial Fractions:

re-write a rational function as a sum of rational functions that are easy to integrate.

$\frac{\text{polynomial}}{\text{polynomial}}$

(just algebra!)

1st Case: Denominator — Repeated Linear Factors

$$\frac{\text{numerator}}{(ax+r)^n} = \frac{C_1}{ax+r} + \frac{C_2}{(ax+r)^2} + \frac{C_3}{(ax+r)^3} + \dots + \frac{C_n}{(ax+r)^n}$$

numerator: polynomial, degree $< n$

a, r const

n natural

(ex) $$\int \underbrace{\frac{6x+7}{4x^2+20x+25}}_{\text{rational}}\,dx$$

denominator: $(2x+5)^2$

$$\frac{6x+7}{(2x+5)^2} = \underbrace{\frac{C}{(2x+5)} + \frac{D}{(2x+5)^2}}_{\text{easier to }\int}$$

Find C, D

$$= \frac{C(2x+5)+D}{(2x+5)^2}$$

common demominator: $(2x+5)^2$

$$\underbrace{6}x+\underbrace{7} = C(2x+5)+D = \underbrace{(2C)}x + \underbrace{(5C+D)}$$

$6 = 2C \longrightarrow \boxed{C=3}$

and $7 = 5C+D$

$7 = 5\cdot 3+D$

$= 15+D$

so $\boxed{D=-8}$

$$\int \frac{6x+7}{(2x+5)^2}\,dx = \int \frac{3}{2x+5} + \frac{-8}{(2x+5)^2}\,dx = \int \left[\frac{3}{u} - \frac{8}{u^2}\right]\cdot\frac{1}{2}\cdot du$$

$u = 2x+5$

$du = 2dx$

$\frac{1}{2}du = dx$

$$= \frac{1}{2}\int \frac{3}{u} - 8u^{-2}\,du = \frac{1}{2}\left[3\ln|u| + 8u^{-1}\right] + C$$

$$= \boxed{\frac{1}{2}\left[3\ln|2x+5| + \frac{8}{2x+5}\right] + C}$$

ex $\int \frac{x^2+6x+10}{(x+3)^3}\,dx$

$$\frac{x^2+6x+10}{(x+3)^3} = \frac{C}{(x+3)} + \frac{D}{(x+3)^2} + \frac{E}{(x+3)^3}$$

Find C, D, E

$$= \frac{C(x+3)^2 + D(x+3) + E}{(x+3)^3}$$

Common Denominator $(x+3)^3$

$$x^2+6x+10 = C(x+3)^2 + D(x+3) + E$$

if $x=-3$:

$$9-18+10 = 0+0+E$$

$\boxed{1=E}$

$\boxed{C=1}$

$$x^2+\underline{6}x+\underline{10} = C(x^2+6x+9) + D(x+3) + 1$$

$$= x^2(\underline{C}) + x(6C+D) + (9C+3D+1)$$

$$= x^2 + x\underline{(6+D)} + \underline{(10+3D)}$$

$6+D=6$

$\boxed{D=0}$

$$\int \frac{x^2+6x+10}{(x+3)^3}\,dx = \int \frac{1}{x+3} + \frac{1}{(x+3)^3}\,dx = \text{etc.}$$

Case 2: Denom has distinct linear factors

all differant
no 2 same

Rule: $$\frac{\text{num}}{(a_1x+r_1)(a_2x+r_2)\cdots(a_nx+r_n)} = \frac{A}{(a_1x+r_1)} + \frac{B}{(a_2x+r_2)} + \cdots + \frac{C}{(a_nx+r_n)}$$

a_i, r_i const
num: polynomial, degree $< n$
a_ix+r_i all differant

ex $\displaystyle\int \frac{7x+13}{2x^2+x-10}\,dx$

$$\frac{7x+13}{(2x+5)(x-2)} = \frac{A}{2x+5} + \frac{B}{x-2}$$

Find A, B

$$= \frac{A(x-2) + B(2x+5)}{(2x+5)(x-2)}$$

common denom

$$\underbrace{7}x + \underbrace{13} = Ax - 2A + 2Bx + 5B$$

$$= x\underbrace{(A+2B)} + \underbrace{(-2A+5B)}$$

$7 = A + 2B \rightarrow A = 7 - 2B$

$13 = -2A + 5B$

$13 = -2(7-2B) + 5B$

$13 = -14 + 4B + 5B$

$27 = 9B$

$\boxed{B=3}$

$A = 7 - 2(3)$

$\boxed{A=1}$

$$\int \frac{7x+13}{(2x+5)(x-2)}\,dx = \underbrace{\int \frac{1}{2x+5} + \frac{3}{x-2}\,dx}_{\text{easier}}$$

Case 3: Some distinct, some repeated linear factors in denom

ex $$\frac{4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \quad \text{etc.}$$

Possible Complication: deg of num $\geq$ deg of denom

In this case: DIVIDE

ex: $$\frac{13}{3} = \frac{12+1}{3} = \frac{4\cdot 3+1}{3} = \frac{4\cdot\cancel{3}}{\cancel{3}} + \frac{1}{3} = 4+\frac{1}{3}$$

pulled out biggest multiple of denom from num

separate fraction + cancel

ex: $$\frac{x^2+4x-7}{x^2-3x+2} = \frac{x^2-3x+2 \ +7x-9}{x^2-3x+2} = \frac{x^2-3x+2}{x^2-3x+2} + \frac{7x-9}{x^2-3x+2}$$

deg of num = deg of denom so partial fractions won't work (yet)

$$= 1 + \underbrace{\frac{7x-9}{x^2-3x+2}} = 1+\frac{7x-9}{(x-1)(x-2)}$$ $\boxed{\text{etc...}}$

deg of num < deg of denom: can do partial fractions ex

$$\frac{x^3 - 3x^2 + 9x - 9}{x^2 - 3x + 2} = \frac{x^3 - 3x^2 + 2x \;\; + 7x - 9}{x^2 - 3x + 2}$$

Can't do partial fractions yet - deg of num too big

$$= \frac{x^3 - 3x^2 + 2x}{x^2 - 3x + 2} + \frac{7x - 9}{x^2 - 3x + 2}$$

Note: $x(x^2 - 3x + 2) = x^3 - 3x^2 + 2x$

$$= \frac{x\cancel{(x^2 - 3x + 2)}}{\cancel{x^2 - 3x + 2}} + \frac{7x - 9}{x^2 - 3x + 2}$$

$$= \underbrace{x + \frac{7x - 9}{x^2 - 3x + 2}}_{\text{now: can do partial fractions}}$$

ex $\dfrac{2x^3-5x^2+8x-7}{x^2-3x+2} = \dfrac{2x^3-6x^2+4x + x^2+4x-7}{x^2-3x+2}$

Note: $2x(x^2-3x+2)$

$= 2x^3-6x^2+4x$

$= \dfrac{2x\cancel{(x^2-3x+2)}}{\cancel{x^2-3x+2}} + \dfrac{x^2+4x-7}{x^2-3x+2}$

$= 2x + \dfrac{x^2+4x-7}{x^2-3x+2} = 2x + \dfrac{x^2-3x+2 \; +7x-9}{x^2-3x+2}$

Note: x^2-3x+2

$= 2x + \dfrac{x^2-3x+2}{x^2-3x+2} + \dfrac{7x-9}{x^2-3x+2}$

$= 2x+1+\underbrace{\dfrac{7x-9}{x^2-3x+2}}_{\text{can do part frac}}$

All suggested HW through 7.5

Ch 7.7: Numerical Integration

Motivation: sometimes we can't (don't want to) find antiderivative

Recall: $\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$

ex $\int e^{x^2}\,dx$

ex $\int \frac{1}{\ln x}\,dx$

ex $\int \sin(x^2)\,dx$

Absolute vs Relative Error

Absolute Error:

$|\text{exact} - \text{approx}|$

Relative Error:

$\dfrac{\text{abs error}}{|\text{actual}|}$

Case 1: 500g sack of flour mistakenly labeled 495g

$|500 - 495| = 5g$

$\dfrac{5}{500} = \dfrac{1}{100} : 1\%$

Case 2: 5g bottle of medicine mistakenly labeled 10g

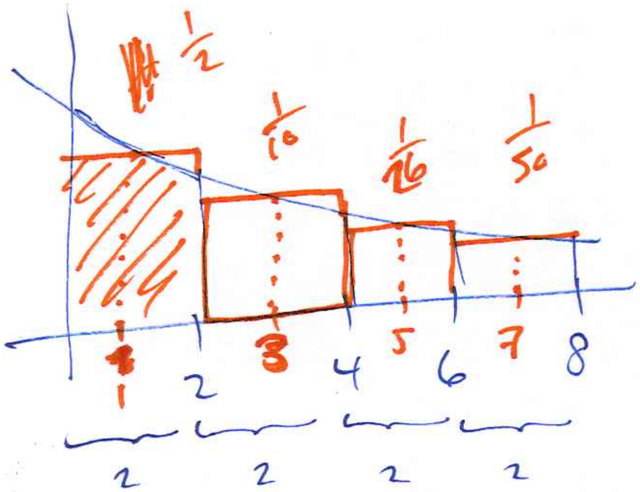
$|5 - 10| = 5g$

$\dfrac{5}{5} = 1 : 100\%$

ex We already saw midpt Riemann Sums
If we take n intervals (~~not~~ limit)
"Midpt Approximation"

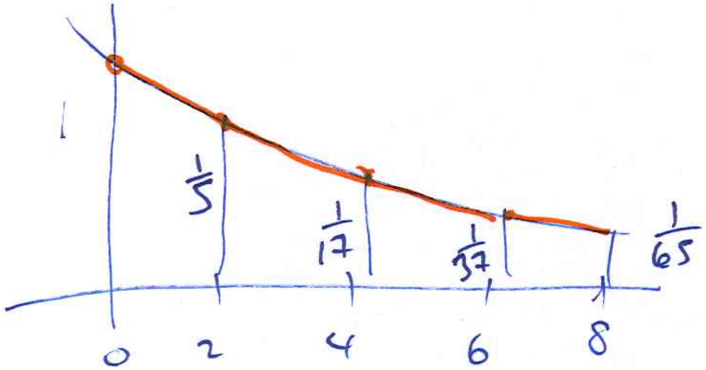
ex) approx $\int_0^8 \frac{1}{1+x^2}\,dx$ using midpt approx, $n=4$

$$\approx 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{10}\right) + 2\left(\frac{1}{26}\right) + 2\left(\frac{1}{50}\right)$$



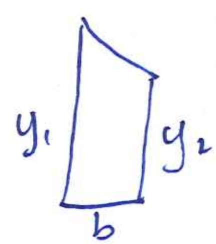
Approx fcn by constant line

ex) $\int_0^8 \frac{1}{1+x^2}\,dx$



Approx fcn using lines

"Trapezoid Rule"



Area of trapezoid:

$\frac{1}{2} b (y_1 + y_2)$

$$\frac{1}{2}(2)\left(1+\frac{1}{5}\right) + \frac{1}{2}(2)\left(\frac{1}{5}+\frac{1}{17}\right) + \frac{1}{2}(2)\left(\frac{1}{17}+\frac{1}{37}\right) + \frac{1}{2}(2)\left(\frac{1}{37}+\frac{1}{65}\right)$$

General Form

Trapezoid: $$\int_a^b f(x)\,dx \approx \Delta x\left(\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right)$$

where $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$

Midpoint: $$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\Delta x$$