Feb 27, 2017

Ch 7.4 : Trig Substitutions

Motivation: $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2+2x+1}}\,dx = \int_3^7 \frac{1}{\cancel{\sqrt{(x+1)^2}}}\,dx$

$\sqrt{(x+1)^2} = |x+1|$

$= x+1$

Trick: cancel and $\sqrt{\ }$

$$= \int_3^7 \frac{1}{x+1}\,dx = \ln(7+1) - \ln(3+1)$$

$$= \ln 8 - \ln 4 = \ln(8/4) = \boxed{\ln 2}$$

Similar Example: $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx$

Goal: write $\sqrt{x^2+1} = \sqrt{(\ \ )^2}$

Recall:

$\tan^2\theta + 1 = \sec^2\theta$

Want to make a substitution so that

x^2+1 becomes

$\tan^2\theta + 1$

$x^2 = \tan^2\theta$

$\boxed{x = \tan\theta}$

Idea: use $x = \tan\theta$

Looking ahead:

$$\sqrt{x^2+1} \underset{x=\tan\theta}{=} \sqrt{\tan^2\theta+1} \underset{\text{trig identity}}{=} \cancel{\sqrt{}}\, \sec^{\cancel{2}}\theta = \sec\theta$$

$\sqrt{}$ cancelled out - good substitution!

Do sub: $x = \tan\theta$
$dx = \sec^2\theta \, d\theta$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sec\theta} \cdot \sec^2\theta \, d\theta = \int \sec\theta \, d\theta \underset{\text{memorized}}{=} \ln|\sec\theta + \tan\theta| + C$$

Get original variable back:

Two Methods

$x = \tan\theta$ (sub)

① Use what we already did

$\sqrt{x^2+1} = \sec\theta$

(works ~~most~~ most, not all, of the time — easiest)

Integral: $\boxed{\ln\left|\sqrt{x^2+1} + x\right| + C}$

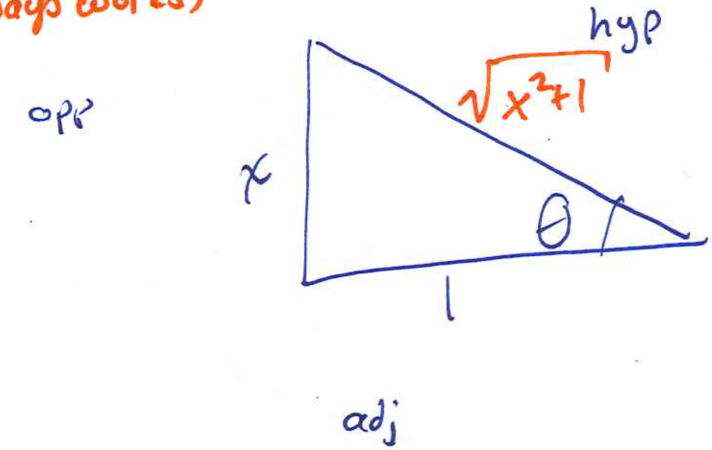
② Use a triangle.

(always works)

Sub: $\tan\theta = x = \frac{x}{1}$ $\qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$

Pythagorus:

$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{x^2+1}}{1}$



Integral: $\boxed{\ln\left|\sqrt{x^2+1} + x\right| + C}$

Identities:

$$1-\sin^2\theta = \cos^2\theta$$

$$1+\tan^2\theta = \sec^2\theta$$

$$\sec^2\theta - 1 = \tan^2\theta$$

constant − function

constant + function

function − constant

Method: (usually: $\sqrt{\text{quadratic}}$)

(1) Simplify using a trig substitution (usually: cancel a $\sqrt{\ }$)

(2) Evaluate resulting trig integral (methods in §7.3)

(3) get original variable back

ex

$$\int \frac{1}{(x^2-16)^{3/2}}\,dx = \int \frac{1}{\sqrt{x^2-16}^{\,3}}\,dx$$

$\sqrt{\text{quad}}$

quadratic fcn:

x^2-16

form: function − constant

what sub will help?

$\sec^2\theta - 1 = \tan^2\theta$ most similar identity

$16\sec^2\theta - 16 = 16\tan^2\theta$ make constants match

want a sub. that will make

x^2-16 become

$16\sec^2\theta - 16$

Need: $x^2 = 16\sec^2\theta$

$\boxed{x = 4\sec\theta}$ substitution

Check we chose a good substitution:

$$\sqrt{x^2-16} = \sqrt{16\sec^2\theta-16} = \sqrt{16(\sec^2\theta-1)}$$

$x=4\sec\theta$

$$= \cancel{\sqrt{}}16\tan^{\cancel{2}}\theta = 4\tan\theta$$

$\sqrt{}$ cancelled: good substitution!

Actually do substitution: $x=4\sec\theta$

$dx = 4\sec\theta\tan\theta\, d\theta$

$$\int \frac{1}{\sqrt{x^2-16}^3}\,dx = \int \frac{1}{(4\tan\theta)^3}\,4\sec\theta\tan\theta\,d\theta$$

Evaluate $\int$:

$$" \quad \frac{1}{4^{\cancel{3}\,2}}\int \frac{1}{\tan^{\cancel{3}\,2}\theta}\cdot\cancel{4}\sec\theta\,\cancel{\tan\theta}\,d\theta$$

$$\frac{1}{16}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{16}\int\left[\frac{1}{\cos\theta} \Big/ \left(\frac{\sin\theta}{\cos\theta}\right)^2 \right] d\theta$$

$$= \frac{1}{16}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{16}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta \;=\; \frac{1}{16}\int \frac{1}{u^2}\,du = \frac{1}{16}\int u^{-2}\,du$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= \frac{1}{16}\cdot\left(-u^{-1}\right) + C$$

$$= \frac{-1}{16}\cdot\frac{1}{\sin\theta} + C$$

$$= \boxed{\frac{-1}{16}\cdot\frac{x}{\sqrt{x^2-16}} + C}$$

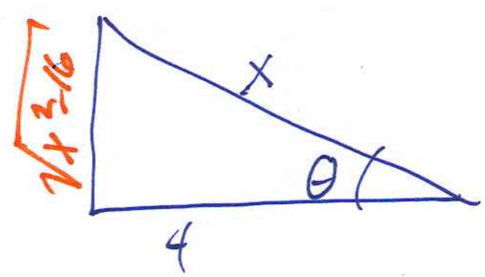
Get variable back to x

$x = 4\sec\theta$

$\frac{x}{4} = \sec\theta = \frac{hyp}{adj}$

Want: $\frac{1}{\sin\theta} = \frac{hyp}{opp}$

$= \frac{x}{\sqrt{x^2-16}}$



ex $\displaystyle\int \frac{\sqrt{4x^2-1}}{x}\,dx$

$\sqrt{\text{quadratic}}$: use trig substitution

Want:

$\sqrt{4x^2-1} = \cancel{\sqrt{(\quad)^2}}$

form: function - constant

Identities:

$1-\sin^2\theta = \cos^2\theta$

$1+\tan^2\theta = \sec^2\theta$

* $\boxed{\sec^2\theta - 1 = \tan^2\theta}$

function - constant

$4x^2-1$

$\sec^2\theta - 1 = \tan^2\theta$

All I need:

$4x^2 = \sec^2\theta$

$x^2 = \frac{1}{4}\sec^2\theta$

$\boxed{x = \frac{1}{2}\sec\theta}$ use to substitute

Check that we chose well:

$$\sqrt{4x^2-1} \underset{x=\frac{1}{2}\sec\theta}{=} \sqrt{4\left(\frac{1}{4}\sec^2\theta\right)-1}$$

$$= \sqrt{\sec^2\theta - 1} = \cancel{\sqrt{\tan^2\theta}} = \tan\theta$$

$\sqrt{}$ cancelled - good sub.

Now: $x = \frac{1}{2}\sec\theta$
$dx = \frac{1}{2}\sec\theta\tan\theta\, d\theta$

(Ch 7.3)

$$\int \frac{\sqrt{4x^2-1}}{x}\,dx = \int \frac{\tan\theta}{\cancel{\frac{1}{2}\sec\theta}} \cdot \cancel{\frac{1}{2}\sec\theta}\tan\theta\, d\theta = \int \tan^2\theta\, d\theta$$

$$= \int (\sec^2\theta - 1)\, d\theta = \tan\theta - \theta + C$$

Get x back:

Already saw $\tan\theta = \sqrt{4x^2-1}$

What is θ?

$$x = \frac{1}{2}\sec\theta$$
$$2x = \sec\theta$$
$$\theta = \text{arcsec}(2x)$$

So: $\tan\theta - \theta + C$

$$= \boxed{\sqrt{4x^2-1} - \text{arcsec}(2x) + C}$$

Technically,

$$\tan\theta = \underbrace{\tan(\text{arcsec}(2x))}_{\text{almost unusable}}$$

Use $\sqrt{4x^2-1}$ instead!

Don't write: trig(arctrig(x)) X