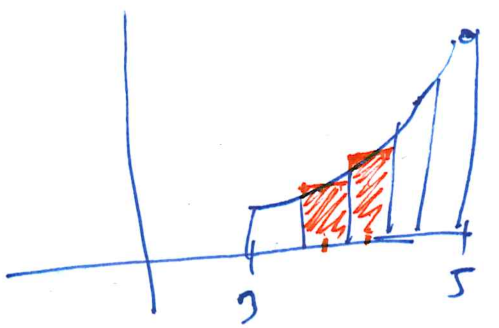
Riemann Sums to Approximate Area Under Curve

ex) Approx area under $y=(x+2)^2$; $[3,5]$, using $n=100$ rectangles, Midpoint Riemann Sum



General Form:

$$\sum_{k=1}^{100} \Delta x \cdot f(a+(k-\tfrac{1}{2})\Delta x), \quad \Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{5-3}{100} = \frac{2}{100} = \frac{1}{50}$$

$a=3$

Sum: $$\sum_{k=1}^{100} \frac{1}{50} \cdot f(\underbrace{3+(k-\tfrac{1}{2})\tfrac{1}{50}})$$

$$= \sum_{k=1}^{100} \frac{1}{50}(\underbrace{3+(k-\tfrac{1}{2})\tfrac{1}{50}} + 2)^2$$

$$\sum_{k=1}^{100} \frac{1}{50}\left(5 + k(1/50) - \frac{1}{2}\left(\frac{1}{50}\right)\right)^2$$

$$= \sum_{k=1}^{100} \frac{1}{50}\left(5 - \frac{1}{100} + \frac{1}{50}k\right)^2$$

$$= \sum_{k=1}^{100} \frac{1}{50}\left(\frac{499}{100} + \frac{1}{50}k\right)^2$$

$$= \sum_{k=1}^{100} \frac{1}{50}\left(\frac{499^2}{100^2} + 2\left(\frac{499}{100}\right)\left(\frac{1}{50}\right)k + \frac{1}{50^2}k^2\right)$$

$$= \sum_{k=1}^{100} \left(\frac{499^2}{100^2 \cdot 50} + \frac{1}{50} \cdot \frac{2 \cdot 499}{100 \cdot 50} k + \frac{1}{50^3} k^2\right)$$

$$= \sum_{k=1}^{100} \frac{499^2}{100^2 \cdot 50} + \sum_{k=1}^{100} \frac{1 \cdot 2 \cdot 499}{50^2 \cdot 100} k + \sum_{k=1}^{100} \frac{1}{50^3} k^2$$

$$= \left(\frac{499^2}{\cancel{100^2} \cdot 50}\right)\cancel{100} + \frac{\cancel{2} \cdot 499}{50^2 \cdot \cancel{100}_{50}} \underbrace{\sum_{k=1}^{100} k} + \frac{1}{50^3}\underbrace{\sum_{k=1}^{100} k^2}$$

Formula:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{499^2}{100 \cdot 50} + \frac{499}{50^3}\left(\frac{100 \cdot 101}{2}\right) + \frac{1}{50^3} \cdot \left(\frac{100 \cdot 101 \cdot 201}{6}\right) = 72.66666$$

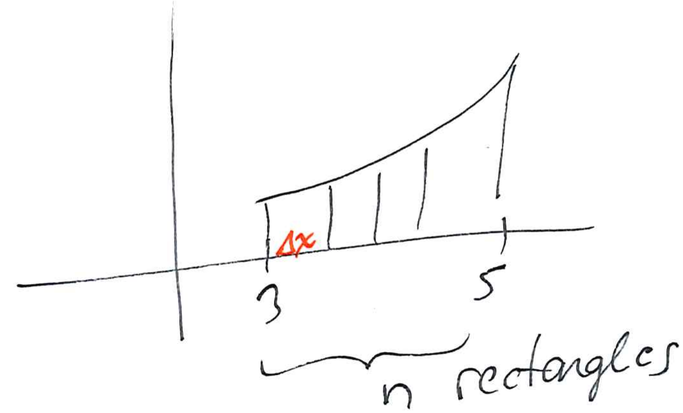
(calculator)

Method for finding exact area under $y = f(x)$

($f(x)$ continuous) on $[a, b]$:

- Take a Rieman Sum with n intervals
  - leave n (not replace it with, e.g. 100)
  - Doesn't matter which Riemann sum (right Riemann sum is easiest - might as well use that one)
- Evaluate using formulas (eg $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$)
- Take $\lim_{n\to\infty}$

(ex) Find exact area under $y = (x+2)^2$, $[3, 5]$, using Riemann Sum.



General form, right RS:

$$\sum_{k=1}^{n} \Delta x\, f(a+k\Delta x)$$

$$\Delta x = \frac{b-a}{n} = \frac{5-3}{n} = \frac{2}{n}$$

$$= \sum_{k=1}^{n} \frac{2}{n} \cdot f(\underbrace{3+k\cdot\frac{2}{n}})$$

$$= \sum_{k=1}^{n} \frac{2}{n} \cdot (\underbrace{3+\frac{2}{n}k} + 2)^2$$

$$= \sum_{k=1}^{n} \frac{2}{n}\left(5+\frac{2}{n}k\right)^2 = \sum_{k=1}^{n} \frac{2}{5}\left(25 + 2\cdot 5\cdot \frac{2}{n}k + \frac{4}{n^2}k^2\right)$$

$$= \sum_{k=1}^{n} \frac{2}{n}\left(25 + \frac{20}{n}k + \frac{4}{n^2}k^2\right) = \sum_{k=1}^{n}\left(\frac{50}{n} + \frac{40}{n^2}k + \frac{8}{n^3}k^2\right)$$

$$= \sum_{k=1}^{n} \frac{50}{n} + \sum_{k=1}^{n} \frac{40}{n^2} k + \sum_{k=1}^{n} \frac{8}{n^3} k^2$$

$$= \left(\frac{50}{\cancel{n}}\right)\cancel{n} + \frac{40}{n^2} \underbrace{\sum_{k=1}^{n} k} + \frac{8}{n^3} \underbrace{\sum_{k=1}^{n} k^2}$$

Formulas:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= 50 + \frac{40}{n^{\cancel{2}}}\left(\frac{\cancel{n}(n+1)}{2}\right) + \frac{\cancel{8}^{\,4}}{n^{\cancel{3}\,2}}\left(\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}_{\,3}}\right)$$

$$= 50 + 20\left(\frac{n+1}{n}\right) + \frac{4}{3} \cdot \frac{(n+1)}{n} \cdot \frac{(2n+1)}{n}$$

$$= 50 + 20\Big(1 + \underset{\to 0}{\frac{1}{n}}\Big) + \frac{4}{3}\Big(1 + \underset{\to 0}{\frac{1}{n}}\Big)\Big(2 + \underset{\to 0}{\frac{1}{n}}\Big)$$

$$\xrightarrow[n \to \infty]{} 50 + 20(1+0) + \frac{4}{3}(1+0)(2+0) = 50 + 20 + \frac{8}{3} = 70 + \frac{8}{3}$$

$$= 70 + 2 + 2/3 = \boxed{72.\overline{66}}$$

Exact area under $y = (x+2)^2$, $[3, 5]$

Exact area under curve $y = f(x)$, over $[a,b]$:

$$\lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} \Delta x \cdot f(x_k^*)}_{\text{Riemann Sum}} \quad , \quad \Delta x = \frac{b-a}{n}$$

left, right, or middle — could be any

ex Exact area under $y = x^3$, on $[3,4]$, using Rieman Sum.

$\Delta x = \frac{4-3}{n} = \frac{1}{n}$

$a = 3$

General Form Right RS (easiest!) $\sum_{k=1}^{n} \Delta x \cdot f(a + k \cdot \Delta x)$

$$= \sum_{k=1}^{n} \frac{1}{n} \cdot f\left(\underbrace{3 + \frac{1}{n}k}\right) =$$

$$= \sum_{k=1}^{n} \frac{1}{n} \cdot \left(3 + \frac{k}{n}\right)^3$$

Recall:

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

1
1 2 1
1 3 3 1

$$= \sum_{k=1}^{n} \frac{1}{n}\left(27 + 3(9)\left(\frac{k}{n}\right) + 3(3)\left(\frac{k}{n}\right)^2 + \left(\frac{k}{n}\right)^3\right)$$

$$= \sum_{k=1}^{n} \frac{1}{n}\left(27 + \frac{27}{n}k + \frac{9}{n^2}k^2 + \frac{1}{n^3}k^3\right)$$

$$= \sum_{k=1}^{n} \left(\frac{27}{n} + \frac{27}{n^2}k + \frac{9}{n^3}k^2 + \frac{1}{n^4}k^3\right)$$

$$= \sum_{k=1}^{n} \frac{27}{n} + \sum_{k=1}^{n} \frac{27}{n^2}k + \sum_{k=1}^{n} \frac{9}{n^3}k^2 + \sum_{k=1}^{n} \frac{1}{n^4}k^3$$

$$= \left(\frac{27}{n}\right)n + \frac{27}{n^2}\underbrace{\sum_{k=1}^{n} k} + \frac{9}{n^3}\underbrace{\sum_{k=1}^{n} k^2} + \frac{1}{n^4}\underbrace{\sum_{k=1}^{n} k^3}$$

FORMULAS (p 340)

$$= 27 + \frac{27}{n^{\cancel{2}1}}\left(\frac{\cancel{n}\cdot(n+1)}{2}\right) + \frac{\cancel{9}^{3}}{n^{\cancel{3}2}}\left(\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}_{2}}\right) + \frac{1}{n^{\cancel{4}2}}\left(\frac{\cancel{n^2}(n+1)^2}{4}\right)$$

$$= 27 + \frac{27}{2}\left(\frac{n+1}{n}\right) + \frac{3}{2}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) + \frac{1}{4}\left(\frac{n+1}{n}\right)^2$$

$$= 27 + \frac{27}{2}\Big(1+\underbrace{\tfrac{1}{n}}_{\to 0}\Big) + \frac{3}{2}\Big(1+\underbrace{\tfrac{1}{n}}_{\to 0}\Big)\Big(2+\underbrace{\tfrac{1}{n}}_{\to 0}\Big) + \frac{1}{4}\Big(1+\underbrace{\tfrac{1}{n}}_{\to 0}\Big)^2 \xrightarrow[n\to\infty]{} \boxed{27 + \frac{27}{2} + \frac{3}{2}(2) + \frac{1}{4}}$$

exact area