Integration by Parts: One Last Fancy Trick

ex $\int e^x \sin x\,dx = -e^x\cos x - \int -\cos x \cdot e^x\,dx$

$u: e^x \qquad du = e^x\,dx$

$dv: \sin x\,dx \rightarrow v = -\cos x$

$= -e^x\cos x + \int e^x\cos x\,dx$

$u: e^x \qquad du = e^x\,dx$

$dv: \cos x\,dx \rightarrow v = \sin x$

$= -e^x\cos x + e^x\sin x - \int e^x\sin x\,dx$

Now: $\int e^x\sin x\,dx = -e^x\cos x + e^x\sin x - \int e^x\sin x\,dx$

$+\int e^x\sin x\,dx \qquad\qquad +\int e^x\sin x\,dx$

$2\int e^x\sin x\,dx = -e^x\cos x + e^x\sin x$

$$\boxed{\int e^x\sin x\,dx = \frac{1}{2}\left[-e^x\cos x + e^x\sin x\right] + C}$$

(ex) $\int e^{-x}\cos x\,dx = e^{-x}\sin x - \int \sin x\cdot(-e^{-x})\,dx$

$u = e^{-x}$ $\quad du = -e^{-x}dx$

$dv = \cos x\,dx$ $\quad v = \sin x$

$u = e^{-x}$ $\quad du = -e^{-x}dx$

$dv = \sin x\,dx$ $\quad v = -\cos x$

$= e^{-x}\sin x + \int e^{-x}\sin x\,dx$

$= e^{-x}\sin x + (-\cos x)(e^{-x}) - \int (+\cos x)(+e^{-x})\,dx$

$= e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\cos x\,dx$

$+\int e^{-x}\cos x\,dx$

$2\int e^{-x}\cos x\,dx = e^{-x}\sin x - e^{-x}\cos x + C$

$$\boxed{\int e^{-x}\cos x\,dx = \frac{1}{2}\left[e^{-x}\sin x - e^{-x}\cos x\right] + C}$$

Now: all §7.2

§7.3 Trig Integrals

(ex) $\int \underbrace{\sin^2 x}_{u^2} \underbrace{\cos x\, dx}_{du} = \int u^2 du = \frac{1}{3}u^3 + C = \boxed{\frac{1}{3}\sin^3 x + C}$

$u = \sin x$

$du = \cos x\, dx$

(ex) $\int \sin^4 x \cos^5 x\, dx = \int \underbrace{\sin^4 x}_{u^4} \cdot \underbrace{\cos^4 x}_{?} \cdot \underbrace{\cos x\, dx}_{du}$

MAYBE: $u = \sin x$

$du = \cos x\, dx$

$= \int \sin^4 x \cdot (\cos^2 x)^2 \cdot \cos x\, dx$

Recall: $\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

$(\cos^2 x)^2 = (1 - \sin^2 x)^2$

$= \int \underbrace{\sin^4 x}_{u^4} \underbrace{(1 - \sin^2 x)^2}_{(1-u^2)^2} \underbrace{\cos x\, dx}_{du}$

$= \int u^4 (1-u^2)^2 du = \int u^4 (1 - 2u^2 + u^4) du$

$= \int (u^4 - 2u^6 + u^8) du = \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$

$= \boxed{\frac{1}{5}(\sin^5 x) - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C}$

Evaluating $\int \sin^n x \cos^m x\, dx$:

If cosine (sine) has odd power:

- reserve one for du
- change the rest (even #) into sine (cosine)
- $u = \sin x$, $du = \cos x$ $\left(\begin{matrix} u = \cos x \\ du = \sin x \end{matrix}\right)$

ex) $\int \sin^3 x \cos^\pi x\, dx = \int \sin^2 x \cos^\pi x \underbrace{\sin x\, dx}$

$= \int \underbrace{(1-\cos^2 x)} \underbrace{\cos^\pi x}_{u^\pi} \underbrace{\sin x\, dx}_{-du} = \int -(1-u^2) u^\pi\, du$

$u = \cos x$
$du = -\sin x\, dx$
$-du = \sin x\, dx$

$= \int \left(-u^\pi + u^{\pi+2}\right) du = -\dfrac{u^{\pi+1}}{\pi+1} + \dfrac{u^{\pi+3}}{\pi+3} + C$

$= \boxed{\dfrac{-1}{\pi+1} \cos^{\pi+1} x + \dfrac{1}{\pi+3} \cos^{\pi+3} x + C}$

If cosine & sine both have even powers:

$$\sin^2 x = \frac{1-\cos(2x)}{2} \quad , \quad \cos^2 x = \frac{1+\cos(2x)}{2}$$

ex $\int \sin^2 x \cos^2 x \, dx = \int \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right) dx$

$= \frac{1}{4}\int (1-\cos(2x))(1+\cos(2x))\, dx = \frac{1}{4}\int 1-\cos^2(2x)\, dx$ ← still only even powers

$= \frac{1}{4}\int 1-\left(\frac{1+\cos(4x)}{2}\right) dx = \frac{1}{4}\int 1-\frac{1}{2}-\frac{1}{2}\cos(4x)\, dx$

$= \frac{1}{4}\int \frac{1}{2}-\frac{1}{2}\cos(4x)\, dx = \frac{1}{8}\int 1-\cos(4x)\, dx$

$= \boxed{\frac{1}{8}\left[x - \frac{1}{4}\sin(4x)\right] + C}$

Evaluating $\int \sec^n x \cdot \tan^m x \, dx$

Recall: $\tan^2 x + 1 = \sec^2 x$

$\int \tan x \, dx = \ln|\sec x| + C$

$\int \sec x \, dx = \ln|\sec x + \tan x| + C$

If secant has an even power:

- Reserve $\sec^2 x$ for du
- Turn all other sec $\rightarrow$ tan
- $u = \tan x$
  $du = \sec^2 x \, dx$

If tangent has an odd power:

- Reserve $\sec x \tan x$ for du
- Turn all other tan $\rightarrow$ sec
- $u = \sec x$
  $du = \sec x \tan x \, dx$

ex $\int \sec^4 x \tan^2 x\,dx = \int \sec^2 x \tan^2 x \underbrace{\sec^2 x\,dx}_{du}$

$u = \tan x$
$du = \sec^2 x\,dx$

$= \int \underbrace{(1+\tan^2 x)}_{1+u^2} \underbrace{\tan^2 x}_{u^2} \underbrace{\sec^2 x\,dx}_{du}$

$= \int (1+u^2)\cdot u^2 \cdot du = \int u^2 + u^4\,du = \frac{1}{3}u^3 + \frac{1}{5}u^5 + C$

$= \boxed{\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C}$

ex $\int \sec^3 x \tan^3 x\,dx = \int \sec^2 x \tan^2 x \underbrace{\sec x \tan x\,dx}_{du}$

$u = \sec x$
$du = \sec x \tan x$

$= \int \sec^2 x (\sec^2 x - 1) \underbrace{\sec x \tan x\,dx}$

$= \int u^2(u^2-1)\,du = \int u^4 - u^2\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$

$= \boxed{\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C}$

(ex) $\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x + C$

If secant has an odd power AND tangent has an even power:

NOT IN SYLLABUS
(reduction formula)