Last Time:

$\int \sec^m x \cdot \tan^n x \, dx$:

Even power of secant:

Reserve $\sec^2 x$ for du

Other secants $\to$ tangents

$u = \tan x$

Odd power of tangent:

Reserve $\sec x \tan x$ for du

Other tangents $\to$ secants

$u = \sec x$

Odd power of secant, even power of tangent: reduction formula (we'll skip this)

IDENTITY:

$\tan^2 x + 1 = \sec^2 x$

so

$\tan^2 x = \sec^2 x - 1$

ex $\int \sec^3 x \tan^3 x \, dx = \int \sec^2 x \cdot \underline{\tan^2 x} \cdot \sec x \tan x \, dx$

$= \int \sec^2 x \, \underline{(\sec^2 x - 1)} \cdot \underbrace{\sec x \tan x \, dx}_{du} = \int u^2 (u^2 - 1) \, du$

$u = \sec x$

$du = \sec x \tan x \, dx$

$= \int (u^4 - u^2) \, du = \frac{u^5}{5} - \frac{u^3}{3} + C = \boxed{\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C}$

$\to$ Able to do suggested problems through 7.3

Ch. 7.4 Trig Substitution

Motivation:

$$\int_3^7 \frac{1}{\sqrt{x^2+2x+1}}\,dx = \int_3^7 \frac{1}{\cancel{\sqrt{(x+1)^2}}}\,dx$$

$$= \int_3^7 \frac{1}{x+1}\,dx = \ln|x+1|\Big|_3^7$$

$$= \ln 8 - \ln 4 = \ln(8/4) = \boxed{\ln 2}$$

Nice thing:

$\cancel{\sqrt{(\ )^2}}$

get rid of $\sqrt{\ }$

Very similar integrand

function we're integrating

Goal:

$$\sqrt{x^2+1} = \cancel{\sqrt{(\quad)^{\cancel{2}}}}$$

cancel $\sqrt{\ }$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx$$

Recall: $(\tan\theta)^2+1 = (\sec\theta)^2$

So, if $x=\tan\theta$: $x^2+1 = \tan^2\theta+1 = (\sec\theta)^2$

then: $\sqrt{x^2+1} = \cancel{\sqrt{(\sec\theta)^{\cancel{2}}}} = \sec\theta$

Sub: $x=\tan\theta$

$dx = \sec^2\theta\,d\theta$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sec\theta}\cdot \sec^2\theta\, d\theta = \int \sec\theta\, d\theta$$

(last time – memorize)

$$= \ln\left|\underbrace{\sec\theta}_{\sqrt{x^2+1}} + \underbrace{\tan\theta}_{x}\right| + C$$

$\theta \to x$

$$= \boxed{\ln\left|\sqrt{x^2+1} + x\right| + C}$$

Sub: $x = \tan\theta$

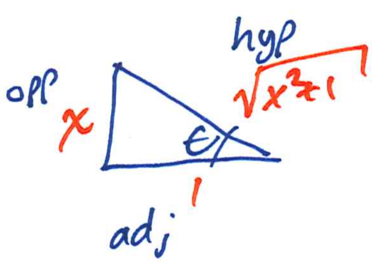
what is $\sec\theta$?

2 WAYS

- 1st Calc: $\sqrt{x^2+1} = \sec\theta$
- Draw a triangle:

$x = \tan\theta$

$\frac{x}{1} = \frac{opp}{adj}$



Then: $\sec\theta = \frac{hyp}{adj} = \frac{\sqrt{x^2+1}}{1}$

So $\sec\theta = \sqrt{x^2+1}$

Idea: $\int \frac{1}{\sqrt{1+x^2}}\,dx = \int (1+x^2)^{-1/2}\,dx = \dots ?$

(ex) $\int x\sqrt{9-x^2}\,dx$

Note: Easier to solve using $u = 9-x^2$
To practice the method, we'll use trig sub

Form: $\sqrt{(\text{quadratic})} \xrightarrow[\text{goal}]{} \cancel{\sqrt{(\quad)^2}}$ get rid of $\sqrt{\ }$

Choose substitution

have $9 - x^2$
const − fcn

$1 - \sin^2\theta = \cos^2\theta$

Closest identity

want $9 - 9\sin^2\theta = 9\cos^2\theta$

fix const

Need: $x^2 = 9\sin^2\theta$

Use: $\boxed{x = 3\sin\theta}$

Identities:

$1 - \sin^2\theta = \cos^2\theta$ const − fcn

$1 + \tan^2\theta = \sec^2\theta$ const + fcn

$\sec^2\theta - 1 = \tan^2\theta$ fcn − const

Check that it's a good idea by looking ahead

($\sqrt{\ } \to$ cancel ?)

$$9-x^2 = 9-(3\sin\theta)^2$$
$$= 9-9\sin^2\theta$$
$$= 9(1-\sin^2\theta)$$
$$= 9\cos^2\theta$$

so: $\sqrt{9-x^2} = \cancel{\sqrt{9\cos^2\theta}}$

$= 3\cos\theta$ $\qquad$ $\sqrt{\ }$ went away – good substitution!

Do substitution: $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$

$$\int x\sqrt{9-x^2}\,dx = \int 3\sin\theta \cdot 3\cos\theta \cdot 3\cos\theta\, d\theta = \int 27 \cdot \cos^2\theta \sin\theta\, d\theta$$

$u = \cos\theta$

$-du = \sin\theta\, d\theta$

Evaluate:

$$-27\int u^2\,du = -27 \cdot \tfrac{1}{3}u^3 + C$$

Get original variable back

$$-9u^3 + C \underset{\substack{\uparrow \\ u=\cos\theta}}{=} -9\cdot\cos^3\theta + C = -9\cdot\left(\tfrac{1}{3}\sqrt{9-x^2}\right)^3 + C$$

Already did:

$\sqrt{9-x^2} = 3\cos\theta$

so $\frac{1}{3}\sqrt{9-x^2} = \cos\theta$

$$= \frac{-9}{27}(9-x^2)^{3/2} + C = \boxed{\frac{-1}{3}(9-x^2)^{3/2} + C}$$

$$\sqrt{A}^3 = (A^{1/2})^3 = A^{3/2}$$

ex $\int \frac{1}{(x^2-16)^{3/2}}\,dx = \int \frac{1}{\sqrt{x^2-16}^{\,3}}\,dx$

want $\sqrt{\ }$ go away

$1-\sin^2\theta = \cos^2\theta$ const - fcn

$1+\tan^2\theta = \sec^2\theta$ const + fcn

$\sec^2\theta - 1 = \tan^2\theta$ fcn - const ← closest

have: x^2-16

fcn - const

$\sec^2\theta - 1 = \tan^2\theta$

fix constant

want: $16\sec^2\theta - 16 = 16\tan^2\theta$

$x^2 = 16\sec^2\theta$

use $\boxed{x = 4\sec\theta}$ as substitution

Check that $x = 4\sec\theta$ really gets rid of $\sqrt{\ }$

$$
\begin{aligned}
x^2 - 16 &= (4\sec\theta)^2 - 16 \\
&= 16\sec^2\theta - 16 \\
&= 16(\sec^2\theta - 1) \\
&= 16 \cdot \tan^2\theta
\end{aligned}
$$

So: $\sqrt{x^2-16} = \sqrt{16 \cdot \tan^2\theta}$ $= 4\tan\theta$

$\sqrt{\ }$ went away – good substitution!

Do substitution: $x = 4\sec\theta$

$dx = 4\sec\theta\tan\theta\, d\theta$

$$\int \frac{1}{\sqrt{x^2-16}^{\,3}}\,dx = \int \frac{1}{(4\tan\theta)^3}\, 4\sec\theta\tan\theta\, d\theta = \frac{1}{16}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

Evaluate $\frac{1}{16}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{16}\int \frac{1}{\cos\theta}\cdot\left(\frac{\cos\theta}{\sin\theta}\right)^2 d\theta$

$= \frac{1}{16}\int \frac{\overset{du}{\cos\theta}}{\sin^2\theta}\,d\theta \qquad = \frac{1}{16}\int \frac{1}{u^2}\,du = \int u^{-2}\,du$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$= -\frac{1}{16}u^{-1} + C$

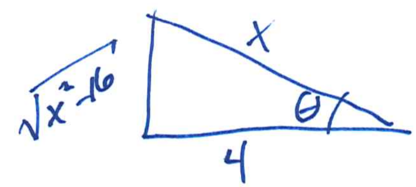
Get original variable back

$= \frac{-1}{16}\cdot(\sin\theta)^{-1} + C = \frac{-1}{16\sin\theta} + C$

$= \boxed{\frac{-1}{16}\cdot\frac{x}{\sqrt{x^2-16}} + C}$

Used: $x = 4\sec\theta$

$\frac{hyp}{adj} = \sec\theta = \frac{x}{4}$



$\sin\theta = \frac{opp}{hyp} = \frac{\sqrt{x^2-16}}{x}$

ex $\int \frac{\sqrt{4x^2-1}}{x}\,dx$

Identities:

$1-\sin^2\theta = \cos^2\theta$

$1+\tan^2\theta = \sec^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

$4x^2-1$

fcn - const

$\sec^2\theta - 1 = \tan^2\theta$

Need: $4x^2 = \sec^2\theta$

$\boxed{2x = \sec\theta}$

Check substitution:

$4x^2-1 = (2x)^2 - 1 = (\sec\theta)^2 - 1 = \tan^2\theta$

So: $\sqrt{4x^2-1} = \cancel{\sqrt{\tan^2\theta}} = \tan\theta$

$\sqrt{\ }$ cancelled: good sub!

Do subst:

$2x = \sec\theta$

$x = 1/2 \sec\theta$

$dx = \frac{1}{2}\sec\theta\tan\theta\, d\theta$

$$\int \frac{\sqrt{4x^2-1}}{x}\,dx$$

$$= \int \frac{\tan\theta}{\cancel{\frac{1}{2}\sec\theta}}\,\cancel{\tfrac{1}{2}\sec\theta}\tan\theta\, d\theta = \int \tan^2\theta\, d\theta$$

$$= \int (\sec^2\theta - 1)\, d\theta = \tan\theta - \theta + C$$

used:

$x = \frac{1}{2}\sec\theta$

$2x = \sec\theta$

$\text{arc}\sec(2x) = \theta$

$\tan\theta = \sqrt{4x^2-1}$

Need to get x back:

$$= \boxed{\sqrt{4x^2-1} - \text{arcsec}(2x) + C}$$

Completing the Square

ex $\displaystyle\int \frac{1}{\sqrt{3-x^2+2x}}\,dx$

3 pieces

$\downarrow$ complete $\square$

2 pieces

$\downarrow$ trig

1 piece $\cancel{\sqrt{(\ )^2}}$

$$3-x^2+2x =$$

$$-[\underbrace{x^2-2x}-3]$$

$$= -[\underbrace{x^2-2x+1}\underbrace{-1-3}]$$

$$= -[(x-1)^2-4]$$

$$= 4-(x-1)^2$$

Recall:

$$(x+a)^2 = \underbrace{x^2+2ax}+a^2$$

$a=-1$

$$(x-1)^2 = x^2-2x+1$$

Choose sub:

const − fcn

$$1 - \sin^2\theta = \cos^2\theta$$

have: $4 - (x-1)^2$

$$1 - \sin^2\theta = \cos^2\theta$$

match constants $4 - 4\sin^2\theta = 4\cos^2\theta$

Need: $(x-1)^2 = 4\sin^2\theta$

$$\boxed{x - 1 = 2\sin\theta}$$

Also can say

$$\boxed{x = 1 + 2\sin\theta}$$

Check our sub:

$$3-x^2+2x = 4-(x-1)^2$$
$$= 4-(2\sin\theta)^2$$
$$= 4-4\sin^2\theta$$
$$= 4(1-\sin^2\theta)$$
$$= 4\cos^2\theta$$

$x = 1+2\sin\theta$
$dx = 2\cos\theta\, d\theta$

Then: $\sqrt{3-x^2+2x} = \sqrt{4\cos^2\theta} = 2\cos\theta$

$\sqrt{}$ gone!

$$\int \frac{1}{\sqrt{3-x^2+2x}}\,dx = \int \frac{1}{2\cos\theta}\cdot 2\cos\theta\, d\theta = \int 1\, d\theta = \theta + C$$

$$= \boxed{\arcsin\left(\frac{x-1}{2}\right) + C}$$

$x-1 = 2\sin\theta$

$\frac{x-1}{2} = \sin\theta$

$\theta = \arcsin\left(\frac{x-1}{2}\right)$

→ Suggested Prob: $\boxed{\S 7.4}$