Ch 7.3 : Trig Integrals

Integrating functions of the form $\sin^a x \cos^b x$ and $\sec^a x \tan^b x$

Recall: (1) $\sin^2 x + \cos^2 x = 1$

(2) $\tan^2 x + 1 = \sec^2 x$

(3) $\sin^2 x = \dfrac{1-\cos(2x)}{2}$

(4) $\cos^2 x = \dfrac{1+\cos(2x)}{2}$

Remark: If you forget (2), you can get it from (1):

$$\frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

ex) $$\int \underbrace{\sin^{10} x}_{u^{10}}\, \underbrace{\cos x\, dx}_{du} = \int u^{10}\, du = \frac{1}{11}u^{11} + C = \boxed{\frac{1}{11}\sin^{11} x + C}$$

$u = \sin x$

$\dfrac{du}{dx} = \cos x$

$du = \cos x\, dx$

ex $\int \sin^{10}x \cdot \cos^5 x\,dx = \int \underbrace{\sin^{10}x}_{u^{10}} \cdot \underbrace{\cos^4 x}_{?} \cdot \underbrace{\cos x\,dx}_{du}$

need to change to sines

Idea: $u = \sin x$
$du = \cos x\,dx$

Identity: $\sin^2 x + \cos^2 x = 1$
so $\cos^2 x = 1 - \sin^2 x$
$\cos^4 x = (\cos^2 x)^2 = (1-\sin^2 x)^2$

$$= \int \underbrace{\sin^{10}x}_{u^{10}} \cdot \underbrace{(1-\sin^2 x)^2}_{(1-u^2)^2} \underbrace{\cos x\,dx}_{du} = \int u^{10}(1-u^2)^2\,du$$

$$= \int u^{10}(1-2u^2+u^4)\,du = \int (u^{10} - 2u^{12} + u^{14})\,du = \frac{1}{11}u^{11} - 2\frac{1}{13}u^{13} + \frac{1}{15}u^{15} + C$$

$$= \boxed{\frac{1}{11}\sin^{11}x - \frac{2}{13}\sin^{13}x + \frac{1}{15}\sin^{15}x + C}$$

ex $\int \sin^5 x \cos^4 x \, dx$

Idea: $\int \sin^5 x \cdot (\cos^2 x)^2 dx$

$= \int \sin^5 x (1-\sin^2 x)^2 dx$

only sine: $u = \sin x$

$du = \cos x dx \leftarrow ??$

Another idea:

$\int \sin^4 x \cos^4 x \sin x dx$

$= \int (\sin^2 x)^2 \cos^4 x \sin x dx$

$= \int (1-\cos^2 x)^2 \cos^4 x \underbrace{\sin x dx}_{-du}$

$u = \cos x$

$du = -\sin x dx$

$-du = \sin x dx$

$-\int (1-u^2)^2 u^4 du = -\int (1-2u^2+u^4)u^4 du = -\int u^4 - 2u^6 + u^8 du$

$= -\left(\frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C\right) = \frac{-1}{5}u^5 + \frac{2}{7}u^7 - \frac{1}{9}u^9 + C$

$= \boxed{\frac{-1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + C}$

General Idea:

$$\int \sin^a x \cos^b x \, dx :$$

If a power is odd, reserve one of them as dx

So, u: use other

ex) $\int \sin^{17} x \cos^{16} x \, dx = \int \underbrace{\sin^{16} x}_{\text{convert to cosine}} \cos^{16} x \underbrace{\sin x \, dx}_{-du}$

so: $u = \cos x$

ex $\displaystyle\int \sin^{2.71} x \cdot \cos^3 x \, dx = \int \sin^{2.71} x \cdot \underbrace{\cos^2 x} \cdot \underbrace{\cos x \, dx}_{du}$

$u : \sin x$

$$= \int \sin^{2.71} x \cdot (1 - \sin^2 x) \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int u^{2.71} (1 - u^2) \, du = \int \left(u^{2.71} - u^{4.71} \right) du$$

$$= \frac{u^{3.71}}{3.71} - \frac{u^{5.71}}{5.71} + C$$

$$= \boxed{\frac{(\sin x)^{3.71}}{3.71} - \frac{(\sin x)^{5.71}}{5.71} + C}$$

ex $\int \sin^5 x \, dx = \int \sin^4 x \underbrace{\sin x \, dx}_{\text{"du"}}$

$u: \cos x$

$= \int (\sin^2 x)^2 \sin x \, dx = \int (1-\cos^2 x)^2 \sin x \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$= \int (1-u^2)^2 (-1) \, du$ $\boxed{\text{etc}}$

What if powers are even?

ex $\int \sin^2 x\,dx$

use half-angle formulas:

$\sin^2 x = \frac{1-\cos 2x}{2}$ $\qquad$ $\cos^2 x = \frac{1+\cos 2x}{2}$

$$\int \sin^2 x\,dx = \int \frac{1-\cos(2x)}{2}\,dx = \frac{1}{2}\int 1-\cos(2x)\,dx$$

$u = 2x$
$du = 2\,dx$
$\frac{1}{2}du = dx$

$$= \frac{1}{2}\int [1-\cos(u)]\,\frac{1}{2}\cdot du$$

$$= \frac{1}{4}\int 1-\cos u\,du = \frac{1}{4}\left[u - \sin u\right] + C$$

$$= \boxed{\frac{1}{4}\left[2x - \sin(2x)\right] + C}$$

ex $\int \sin^2 x \cos^2 x \, dx$

Recall:

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{4}\int (1-\cos 2x)(1+\cos 2x)\, dx = \frac{1}{4}\int \left[1-\cos^2(2x)\right] dx$$

$$\cos^2(2x) = \frac{1+\cos(4x)}{2}$$

$$= \frac{1}{4}\int \left[1 - \frac{1+\cos(4x)}{2}\right] dx$$

$$= \frac{1}{4}\int \left(1-\frac{1}{2}-\frac{1}{2}\cos(4x)\right) dx$$

$$= \frac{1}{4}\int \left(\frac{1}{2}-\frac{1}{2}\cos(4x)\right) dx$$

$$= \boxed{\frac{1}{4}\left[\frac{1}{2}x - \frac{1}{8}\sin(4x)\right] + C}$$

Idea:

last ex

$$\frac{1}{2}\int \cos^2 u \, du = \frac{1}{2}\int (1-\sin^2 u)\, du$$

$\cancel{u = \sin u}$

$\cancel{du = \cos u \, du}$???

Products of Secants and Tangents

Products of Secants and Tangents

$$\int \tan x\,dx = \ln|\sec x| + C$$

The antiderivative of the tangent function is the natural log of the absolute value of the secant function, plus any constant

$$\int \sec x\,dx = \int \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx = \int \frac{\overbrace{\sec^2 x + \sec x \tan x\;dx}^{du}}{\underbrace{\sec x + \tan x}_{u}}$$

$u = \sec x + \tan x$

$\frac{du}{dx} = \sec x \tan x + \sec^2 x$

$du = [\sec^2 x + \sec x \tan x]\,dx$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \boxed{\ln|\sec x + \tan x| + C}$$ ← memorize

(ex) $\int \sec^2 x \tan x \, dx$

1 way: $u = \tan x$ $\quad \int u \, du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\tan^2 x + C}$

$du = \sec^2 x \, dx$

Reserve $\boxed{2}$ secants (not one)

secant: even power

Another way: $u = \sec x$

$du = \sec x \tan x \, dx$ $\quad \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\sec^2 x + C}$

$\int \sec x \cdot \underbrace{\sec x \tan x \, dx}_{du} = \int u \, du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\sec^2 x + C}$

Note: $\boxed{\frac{1}{2}\tan^2 x} + C = \frac{1}{2}(\sec^2 x - 1) + C = \boxed{\frac{1}{2}\sec^2 x - \frac{1}{2}} + C$

$= \frac{1}{2}\sec^2 x + C$ arbitrary constant

Reserve: $\sec x \tan x$ as dx

Odd power of tangent

$u = \sec x$