Substitution Rule with Definite Integrals

ⓔⓧ $\displaystyle\int_0^{\pi/4} \underbrace{\sin^5\theta}_{u^5}\,\underbrace{\cos\theta\, d\theta}_{du} = \int_0^{1/\sqrt{2}} u^5\,du = \frac{1}{6}u^6\Big|_0^{1/\sqrt{2}}$

$u = \sin\theta$

$\frac{du}{d\theta} = \cos\theta$

$du = \cos\theta\, d\theta$

If $\theta = 0$, $u = \sin(0) = 0$

If $\theta = \pi/4$, $u = \sin(\pi/4) = 1/\sqrt{2}$

$$= \frac{1}{6}\left(\frac{1}{\sqrt{2}}\right)^6 - \cancel{\frac{1}{6}(0)^6}$$

$$= \frac{1}{6}\cdot\frac{1}{\sqrt{2}^6} = \frac{1}{6}\cdot\frac{1}{2^3} = \frac{1}{6\cdot 8} = \boxed{\frac{1}{48}}$$

Alternate:

$$\int_{\theta=0}^{\theta=\pi/4} u^5\,du = \frac{1}{6}u^6\Big|_{\theta=0}^{\theta=\pi/4} = \frac{1}{6}(\sin\theta)^6\Big|_{\theta=0}^{\theta=\pi/4}$$

$$= \frac{1}{6}(\sin\pi/4)^6 - \frac{1}{6}(\sin 0)^6 = \frac{1}{6}\left(\frac{1}{\sqrt{2}}\right)^6 - \frac{1}{6}\cdot 0^6 = \frac{1}{6}\cdot\frac{1}{\sqrt{2}^6} = \boxed{\frac{1}{48}}$$

$\frac{1}{2}du$

ex $\displaystyle\int_0^{\ln 4} \frac{e^x}{\underbrace{3+2e^x}_{u}}\,dx = \int_5^{11} \frac{1}{2}\cdot\frac{1}{u}\,du$

$u = 3+2e^x \rightarrow$ If $x=0$, $u = 3+2e^0 = 3+2(1) = 5$

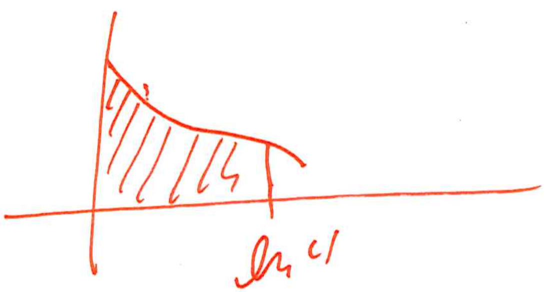
$\frac{du}{dx} = 2e^x$ If $x = \ln 4$, $u = 3+2\cancel{e}^{\cancel{\ln}4} = 3+2(4) = 11$

$du = 2e^x dx$

$\frac{1}{2}du = e^x dx$

$$= \frac{1}{2}\ln|u| \Big|_5^{11} = \frac{1}{2}\ln(11) - \frac{1}{2}\ln(5)$$

$$= \boxed{\frac{1}{2}\ln(11/5)}$$



ex $\displaystyle\int_5^{10} \frac{8t+6}{2t^2+3t}\,dt = \int_{65}^{230} 2\cdot\frac{1}{u}\cdot du$

2du

u

$u = 2t^2+3t$

$\frac{du}{dt} = 4t+3$

$du = (4t+3)\,dt$

$2du = (8t+6)\,dt$

if $t=5$, then $u = 2\cdot 5^2 + 3\cdot 5$
$= 50+15$
$= 65$

if $t=10$, then $u = 2(10)^2 + 3(10)$
$= 200+30$
$= 230$

$$= 2\ln|u|\Big|_{65}^{230} = 2\ln(230) - 2\ln(65)$$

$$= \boxed{2\ln\left(\frac{230}{65}\right)}$$

Suggested Problems: §5.5

Quiz Wed

Ch 7.2: Integration by Parts

(reversing product rule)

$u(x)$, $v(x)$ differentiable functions

Product Rule:

$$\frac{d}{dx}\{u(x)\cdot v(x)\} = u'(x)\cdot v(x) + u(x)\cdot v'(x)$$

So: $$\int [u'(x)v(x) + u(x)v'(x)]\,dx = u(x)\cdot v(x) + C$$

$$\int u'(x)v(x)\,dx + \underbrace{\int u(x)v'(x)\,dx} = u(x)v(x) + C$$

$$\int \underbrace{u(x)}_{u}\cdot\underbrace{v'(x)dx}_{dv} = \underbrace{u(x)\cdot v(x)}_{uv} - \int \underbrace{v(x)}_{v}\cdot\underbrace{u'(x)dx}_{du} + C$$

Mnemonic: $\boxed{\int u\,dv = uv - \int v\,du}$

$$\int u\,dv = uv - \int v\,du$$

ex $\int \underbrace{x}\,\underbrace{\sin x}\,dx$

$u : x$ $\qquad$ $du : 1\,dx$

$dv : \sin x\,dx$ $\rightarrow$ $v : -\cos x$

$= -x\cos x - \int(-\cos x)\,dx$

$= -x\cos x + \int \cos x\,dx$

$= \boxed{-x\cos x + \sin x + C}$

Strategy planning

$x \xrightarrow{u} 1$

$x \xrightarrow{dv} \frac{1}{2}x^2$

$\sin x \xrightarrow{u} \cos x$

$\sin x \xrightarrow{dv} -\cos x$

$$\int u\,dv = uv - \int v\,du$$

ex) $\int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \frac{1}{x}\,dx$

$u: \ln x \qquad du: \frac{1}{x}\,dx$

$dv: x\,dx \rightarrow v: \frac{1}{2}x^2$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\left(\frac{1}{2}x^2\right) + C$$

$$= \boxed{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$$

Planning:

$x \xrightarrow{u} 1$

$x \xrightarrow{dv} \frac{1}{2}x^2$

$\ln x \xrightarrow{u} \frac{1}{x}$

$\ln x \xrightarrow{\cancel{dv}} ?$

Check: $\frac{d}{dx}\left\{ \underbrace{\frac{1}{2}x^2}\,\underbrace{\ln x} - \frac{1}{4}x^2 + c \right\}$

$= \frac{1}{2}x^2\left(\frac{1}{x}\right) + \ln x(x) - \frac{1}{2}x$

$= \cancel{\frac{1}{2}x} + x\ln x - \cancel{\frac{1}{2}x}$

$= x\ln x$ ✓

$\int u\,dv = uv - \int v\,du$

(ex) $\int \underbrace{(3t+5)}_{u} \underbrace{\cos t}_{dv}\, dt$

$u: 3t+5 \qquad\qquad du: 3\,dt$

$dv: \cos t\, dt \rightarrow \qquad v: \sin t$

$(3t+5)\sin t - \int 3 \sin t\, dt$

$= (3t+5)\sin t - 3(-\cos t) + C$

$= \boxed{(3t+5)\sin t + 3\cos t + C}$

Planning:

$3t+5 \xrightarrow{u} 3$

$3t+5 \xrightarrow{dv} \frac{3}{2}t^2+5t$

$\cos t \xrightarrow{u} -\sin t$

$\cos t \xrightarrow{dv} \sin t$

$$\int u\,dv = uv - \int v\,du$$

ex $\displaystyle\int \underbrace{x^3}\,\underbrace{\ln x}\,dx$

$u: \ln x \qquad du: \frac{1}{x}\,dx$

$dv: x^3\,dx \rightarrow \qquad v: \frac{1}{4}x^4$

$$\frac{1}{4}x^4\ln x - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\,dx$$

$$= \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3\,dx$$

$$= \frac{1}{4}x^4\ln x - \frac{1}{4}\cdot\frac{1}{4}x^4 + C$$

$$= \boxed{\frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C}$$

Planning:

$x^3 \xrightarrow{u}$

$\quad \xrightarrow{dv}$

$\ln x \xrightarrow{u}$

$\quad \xrightarrow{\cancel{dv}}$???