Riemann Sums

ex) Approx area under $f(x) = (x+2)^2$ on $[3,5]$
using $n=100$ rectangles.
Use right Riemann Sum

General formula for right RS: (n □, interval $[a,b]$)

$$\sum_{k=1}^{n} \Delta x \cdot f(a+k\Delta x) \quad \text{where} \quad \Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{5-3}{100} = \frac{2}{100} = \frac{1}{50}$$

$a = 3 \quad n = 100$

Riemann Sum: $$\sum_{k=1}^{100} \frac{1}{50} f(\underbrace{3+k(\tfrac{1}{50})}_{x}) = \sum_{k=1}^{100} \frac{1}{50}\left(2+3+\frac{1}{50}k\right)^2$$

$f(x) = (x+2)^2$

$$= \sum_{k=1}^{100} \frac{1}{50}\left(5+\frac{1}{50}k\right)^2 = \sum_{k=1}^{100} \frac{1}{50}\left(25 + \underbrace{2(5)\cdot\frac{1}{50}k}_{\frac{1}{5}k} + \frac{1}{50^2}k^2\right)$$

$$= \sum_{k=1}^{100}\left(\frac{1}{2} + \frac{1}{50\cdot 5}k + \frac{1}{50^3}k^2\right)$$

$$= \underbrace{\sum_{k=1}^{100}\frac{1}{2}} + \sum_{k=1}^{100}\frac{1}{50\cdot 5}k + \sum_{k=1}^{100}\frac{1}{50^3}k^2$$

$$= 100\left(\tfrac{1}{2}\right) + \frac{1}{50\cdot 5}\underbrace{\sum_{k=1}^{100}k}_{\text{formula}} + \frac{1}{50^3}\underbrace{\sum_{k=1}^{100}k^2}_{\text{formula}}$$

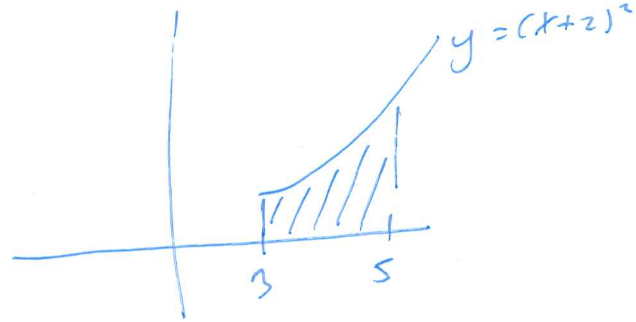
$$\sum_{k=1}^{n}k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= 50 + \frac{1}{50\cdot 5}\cdot\left(\frac{100\cdot 101}{2}\right) + \frac{1}{50^3}\cdot\left(\frac{100\cdot 101\cdot 201}{6}\right)$$

$=_{(\text{calculator})}$ $\boxed{72.9068}$

Approx (right RS) area:



ex) Area under $f(x) = (x+2)^2$, over $[3, 5]$ using $n=100$ rectangles: Midpoint RS

General formula for midpt RS:

$$\sum_{k=1}^{n\ (100)} \Delta x \cdot f\left(a + \left(k - \tfrac{1}{2}\right)\Delta x\right)$$

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{5-3}{100} = \frac{2}{100} = \frac{1}{50}$$

$$a = 3$$

$$\sum_{k=1}^{100} \frac{1}{50} \cdot f\left(3 + \left(k - \tfrac{1}{2}\right)\tfrac{1}{50}\right)$$

$$= \sum_{k=1}^{100} \frac{1}{50} f\left(3 - \tfrac{1}{100} + \tfrac{1}{50}k\right)$$

$$= \sum_{k=1}^{100} \frac{1}{50} \cdot f\left(\tfrac{299}{100} + \tfrac{1}{50}k\right)$$

$$= \sum_{k=1}^{100} \frac{1}{50} \cdot \left(2 + \tfrac{299}{100} + \tfrac{1}{50}k\right)^2$$

$$= \sum_{k=1}^{100} \frac{1}{50}\left(\frac{499}{100} + \frac{1}{50}k\right)^2$$

$$= \sum_{k=1}^{100} \frac{1}{50}\left(\left(\frac{499}{100}\right)^2 + 2\left(\frac{499}{100}\right)\cdot\frac{1}{50}k + \frac{1}{50^2}k^2\right)$$

$$= \sum_{k=1}^{100}\left(\frac{499^2}{100^2\cdot 50} + \frac{2\cdot 499}{100\cdot 50^2}k + \frac{1}{50^3}k^2\right)$$

$$= \underbrace{\sum_{k=1}^{100} \frac{499^2}{100^2\cdot 50}} + \sum_{k=1}^{100} \frac{2\cdot 499}{100\cdot 50^2}k + \sum_{k=1}^{100} \frac{1}{50^3}k^2$$

$$= \cancel{100}\left(\frac{499^2}{100^{\cancel{2}}\cdot 50}\right) + \frac{\cancel{2}\cdot 499}{\cancel{100}_{50}\cdot 50^2}\sum_{k=1}^{100} k + \frac{1}{50^3}\sum_{k=1}^{100} k^2$$

$$= \frac{499^2}{100\cdot 50} + \frac{499}{50^3}\sum_{k=1}^{100} k + \frac{1}{50^3}\sum_{k=1}^{100} k^2$$

$$= \frac{499^2}{100\cdot 50} + \frac{499}{50^3}\left(\frac{100\cdot 101}{2}\right) + \frac{1}{50^3}\cdot\left(\frac{100\cdot 101\cdot 201}{6}\right)$$

Formulas (p 340)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad (n=100)$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

CALCULATOR: 72.6666

(ex) Find exact area under $y=(x+2)^2$; $[3,5]$

Plan: • take RS using n rectangles ← might as well use easiest RS: right RS

• evaluate (in terms of n)

• take limit as $n \to \infty$

$\Delta x = \frac{b-a}{n} = \frac{2}{n}$

General form: $\sum_{k=1}^{n} \Delta x \, f(a + k\Delta x)$

$a = 3$

$$= \sum_{k=1}^{n} \left(\frac{2}{n}\right) \cdot f\left(3 + k \cdot \frac{2}{n}\right) = \sum_{k=1}^{n} \left(\frac{2}{n}\right)\left(2+3+\frac{2}{n}k\right)^2$$

$$= \sum_{k=1}^{n} \left(\frac{2}{n}\right)\left(5+\frac{2}{n}k\right)^2 = \sum_{k=1}^{n} \left(\frac{2}{n}\right)\left(25 + 2(5)\left(\frac{2}{n}k\right) + \left(\frac{2}{n}\right)^2 k^2\right)$$

$$= \sum_{k=1}^{n} \left(\frac{50}{n} + \frac{40}{n^2}k + \left(\frac{2}{n}\right)^3 k^2\right)$$

$$= \sum_{k=1}^{n} \frac{50}{n} + \sum_{k=1}^{n} \frac{40}{n^2} k + \sum_{k=1}^{n} \frac{8}{n^3} k^2$$

$$= \sum_{k=1}^{n} \frac{50}{n} + \frac{40}{n^2} \sum_{k=1}^{n} k + \frac{8}{n^3} \sum_{k=1}^{n} k^2$$

Formulas: (p 340)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= (\cancel{n})\left(\frac{50}{\cancel{n}}\right) + \frac{\overset{20}{\cancel{40}}}{\cancel{n^2}} \cdot \frac{\cancel{n}(n+1)}{\cancel{2}} + \frac{\overset{4}{\cancel{8}}}{n^{\cancel{3}\,2}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{\underset{3}{\cancel{6}}}$$

$$= 50 + 20 \cdot \left(\frac{n+1}{n}\right) + \frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)$$

$$= 50 + 20\left(1 + \frac{1}{n}\right) + \frac{4}{3}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= 50 + 20\left(1 + \frac{1}{n}\right) + \frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

$$\xrightarrow[n \to \infty]{} 50 + 20(1+0) + \frac{4}{3}(1+0)(2+0)$$

$$= 50 + 20 + \frac{8}{3}$$

$$= 70 + 8/3$$

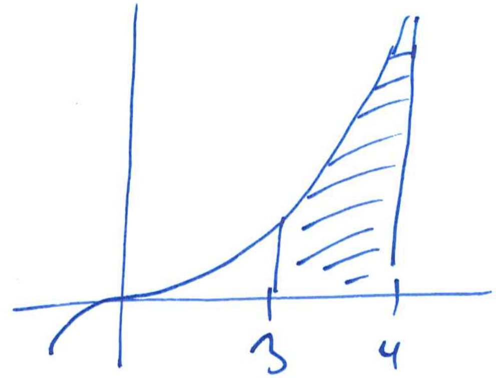
$$= 70 + 2 + 2/3 = \boxed{72 + 2/3} = 72.\overline{66}$$

exact area under $y = (x+2)^2$ from $x = 3$ to $x = 5$

ex Find the exact area under the curve

$y = x^3$

over the interval $[3, 4]$



General Right Riemann Sum:

$$\sum_{k=1}^{n} \Delta x \, f(a + k\Delta x)$$

$$\Delta x = \frac{b-a}{n} = \frac{4-3}{n} = \frac{1}{n}$$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

formulas:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{k=1}^{n} \frac{1}{n} f\left(3 + k\left(\frac{1}{n}\right)\right)$$

$$= \sum_{k=1}^{n} \frac{1}{n} \cdot \left(3 + \frac{k}{n}\right)^3$$

$$= \sum_{k=1}^{n} \frac{1}{n} \left[27 + 3 \cdot 9 \cdot \frac{k}{n} + 3 \cdot 3 \cdot \left(\frac{k}{n}\right)^2 + \left(\frac{k}{n}\right)^3\right]$$

$$= \sum_{k=1}^{n} \frac{1}{n}\left[27 + \frac{27}{n}k + \frac{9}{n^2}k^2 + \frac{1}{n^3}k^3\right]$$

$$= \sum_{k=1}^{n} \left(\frac{27}{n} + \frac{27}{n^2}k + \frac{9}{n^3}k^2 + \frac{1}{n^4}k^3\right)$$

$$= \sum_{k=1}^{n} \frac{27}{n} + \sum_{k=1}^{n} \frac{27}{n^2}k + \sum_{k=1}^{n} \frac{9}{n^3}k^2 + \sum_{k=1}^{n} \frac{1}{n^4}k^3$$

$$= n\left(\frac{27}{n}\right) + \frac{27}{n^2}\sum_{k=1}^{n} k + \frac{9}{n^3}\sum_{k=1}^{n} k^2 + \frac{1}{n^4}\sum_{k=1}^{n} k^3$$

FORMULAS

$$= 27 + \frac{27}{n^{\cancel{2}}}\left(\frac{\cancel{n}(n+1)}{2}\right) + \frac{\cancel{9}^{\,3}}{n^{\cancel{3}\,2}}\left(\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}_{\,2}}\right) + \frac{1}{n^{\cancel{4}\,2}}\left(\frac{\cancel{n^2}(n+1)^2}{4}\right)$$

$$= 27 + \frac{27}{2}\left(\frac{n+1}{n}\right) + \frac{3}{2}\left(\frac{(n+1)}{n} \cdot \frac{(2n+1)}{n}\right) + \frac{1}{4}\left(\frac{n+1}{n}\right)^2$$

$$= 27 + \frac{27}{2}\left(1+\frac{1}{n}\right) + \frac{3}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{1}{4}\left(1+\frac{1}{n}\right)^2$$

$$\xrightarrow[n\to\infty]{} 27 + \frac{27}{2}(1+0) + \frac{3}{2}(1+0)(2+0) + \frac{1}{4}(1+0)^2$$

$$= 27 + \frac{27}{2} + 3 + \frac{1}{4} = 30 + \frac{27}{2} + \frac{1}{4} = 30 + 13 + \frac{1}{2} + \frac{1}{4} = 43 + \frac{3}{4}$$

$$= \boxed{43.75}$$

To find exact area under $y = f(x)$, $[a,b]$:

$$\text{Area} = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \Delta x\, f(x_k^*)\right)$$

could be left / right / MP

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n} \Delta x\, f(a + k\Delta x)\right)$$