Quick Note:

The <u>gradient</u> of a function $f(x,y)$ is the vector

$$\nabla f = \langle f_x , f_y \rangle$$

ex $f(x,y) = x^2+y^2$

$$\nabla f = \langle 2x, 2y \rangle$$

$$\nabla f = \lambda \nabla g$$

$$\Downarrow$$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \end{cases}$$

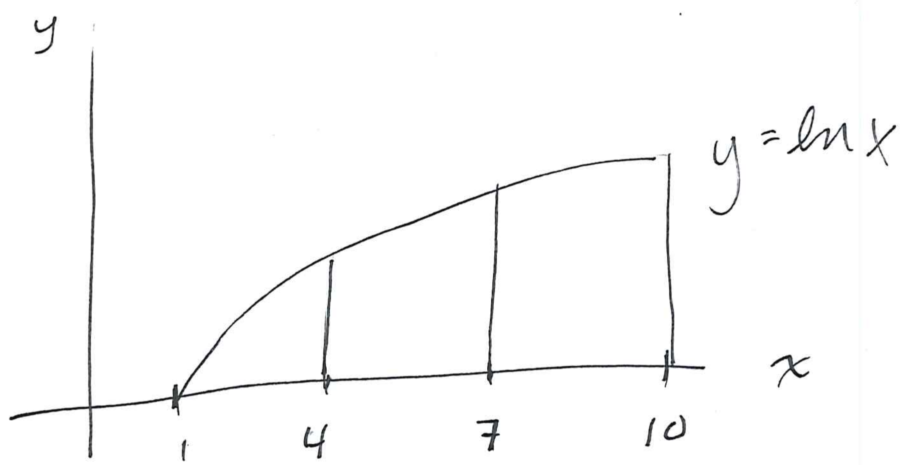
Riemann Sums

ex) Want to approximate area under $y = \ln x$

from $x = 1$ to $x = 10$

using Riemann Sums, 3 subintervals

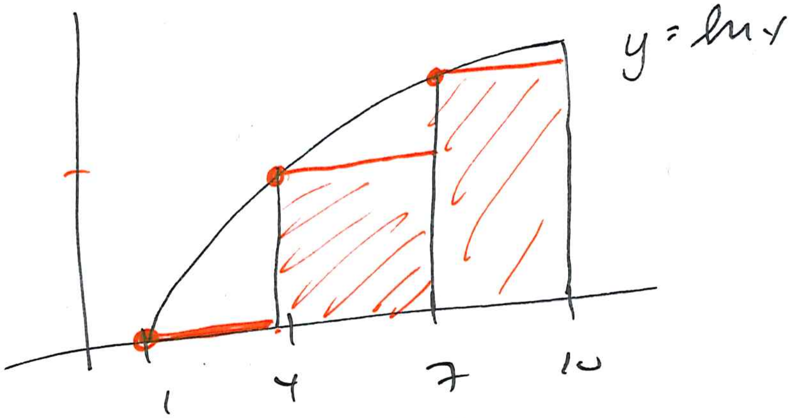
(3 rectangles)



Δx : width of each $\square$

$$\Delta x = \frac{10-1}{3} = 3$$

Left:



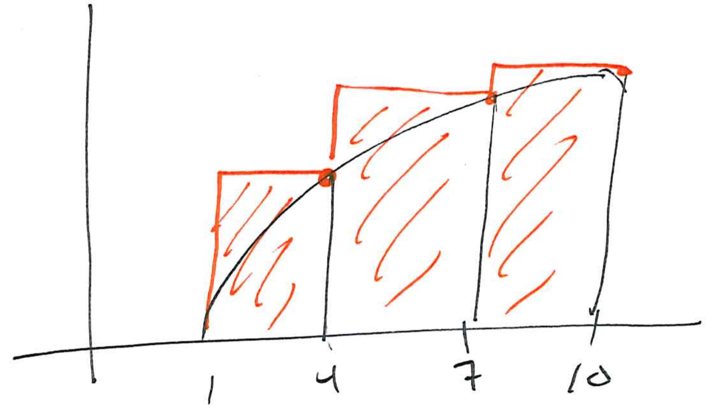
1st ▯: width 3
height 0

2nd ▯: width 3
height ln 4

3rd ▯: width 3
height ln 7

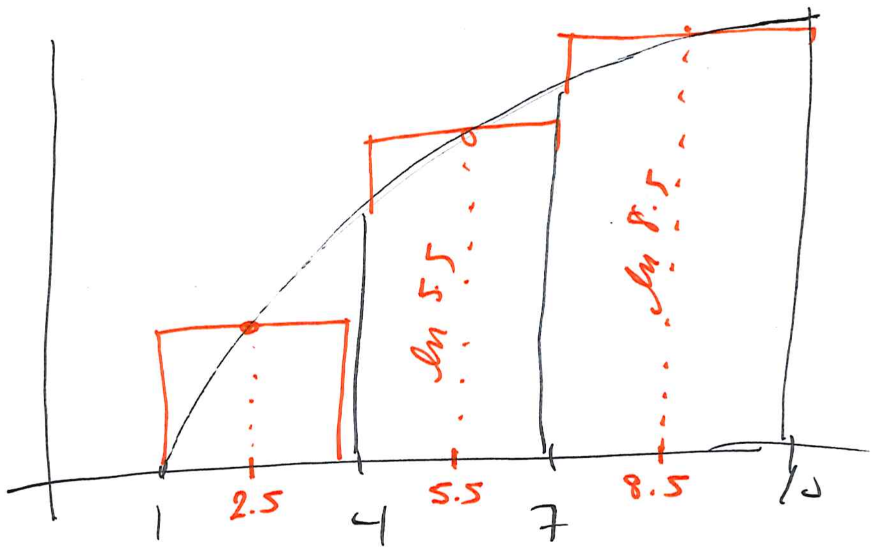
RS: $(3)(0) + (3)\ln 4 + (3)\ln 7$

Right:



RS: $(3)\ln 4 + (3)\ln 7 + 3\ln 10$

Midpoint



RS: $(3)\ln 2.5 + 3\ln 5.5 + 3\ln 8.5$

Review of Σ - notation

$$\sum_{k=a}^{b} f(k)$$

ex: $\displaystyle\sum_{k=-2}^{1} (2k+5) = \boxed{\underset{(k=-2)}{(-4+5)} + \underset{(k=-1)}{(-2+5)} + \underset{(k=0)}{(5)} + \underset{(k=1)}{(2+5)}}$

ex $\displaystyle\sum_{k=6}^{8} (k^2-k) = \boxed{(36-6)+(49-7)+(64-8)}$

Which are OK?

✓ (A) $\sum_{k=1}^{15} (k^2 - k) = \sum_{k=1}^{15} k^2 - \sum_{k=1}^{15} k$

Order doesn't matter in addition

$\underbrace{1^2-1}+\underbrace{2^2-2}+\underbrace{3^2-3} \cdots$

$= (1^2+2^2+3^2+\cdots) - (1+2+3\cdots)$

✓ (B) $\sum_{k=1}^{15} 3k = 3\sum_{k=1}^{15} k$ Factoring

$3(1)+3(2)+3(3)+\cdots+3(15)$

$= 3[1+2+3+\cdots+15]$

$\cancel{\text{(C)}}$ $\sum_{k=1}^{15} k(k-1) \neq k\sum_{k=1}^{15}(k-1)$

$1(0)+2(1)+3(2)+\cdots$

✓ (D) $\sum_{k=1}^{15} (k-1) = -15 + \sum_{k=1}^{15} k$

$\|$

$(1-1)+(2-1)+(3-1)+(4-1)+\cdots+(15-1)$

$\|$

$\underbrace{(1+2+3+4+\cdots+15)}\underbrace{-1-1-1-1\cdots-1}_{15 \text{ times}} = \sum_{k=1}^{15} k - \sum_{k=1}^{15} 1$

$\|$

$\sum_{k=1}^{15} k - 15$

ex Write in Σ-notation:

$$2+3+4+5+6+7 = \sum_{k=2}^{7} k = \sum_{k=1}^{6} (k+1)$$

$$4+6+8+10+12 = \sum_{k=2}^{6} 2k$$

$$= \sum_{k=2}^{6} \cancel{k+2} = 4+\cancel{8}$$

$k=2$ $k=3$

$$5+7+9+11+13 = \sum_{k=2}^{6} (2k+1)$$

$$3.5+6.5+9.5+12.5+15.5 = \sum_{k=1}^{5} \left(3k+\tfrac{1}{2}\right)$$

$$\tfrac{1}{2}+1+2+4+8+16+32 = \sum_{k=-1}^{5} 2^k$$

2^{-1} 2^0 2^1 2^2 2^3 2^4 2^5

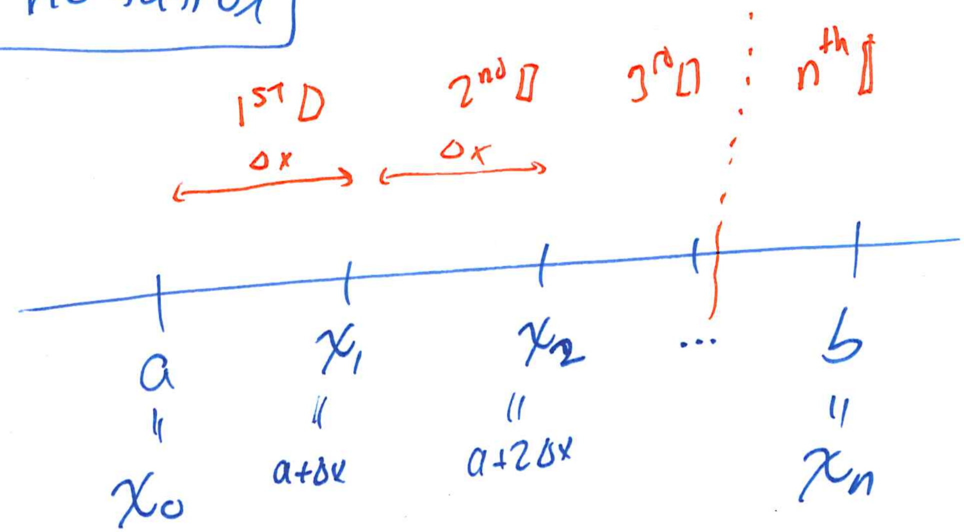
$$-\tfrac{1}{2}+1-2+4-8+16-32 = \sum_{k=-1}^{5} (-2)^k$$

$(-2)^0$ $(-2)^1$ $(-2)^2$ $(-2)^3$

$$\sum_{k=2}^{6} (2k+1) = 5+7+9+11+13$$

Riemann Sums in Σ-notation

function $f(x)$
over $[a, b]$
use n subintervals
(n rectangles)

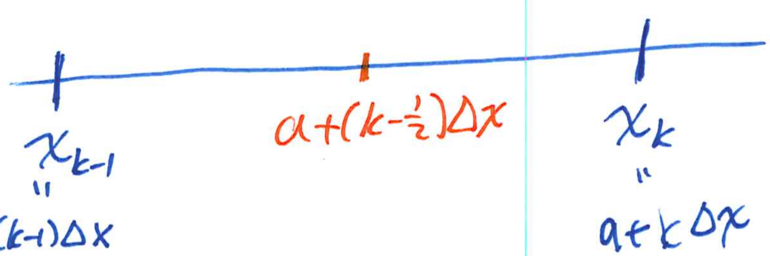


Bases of all rectangles:

$$\Delta x = \frac{b-a}{n}$$

Height: 1) depends on left/right/middle Riemann sum
2) depends on which □

(k^{th} □)



Area of k^{th} $\square$:

$(\text{base})(\text{height})$

$\Delta x\,(\text{height})$

Left: $\Delta x\, f(x_{k-1}) = \Delta x\, f(a+(k-1)\Delta x)$

Right: $\Delta x\, f(x_k) = \Delta x \cdot f(a+k\Delta x)$

Midpt: $\Delta x \cdot f(a+(k-\frac{1}{2})\Delta x)$

where: $\Delta x = \frac{b-a}{n}$

General Formulas for Riemann Sums:

Left: $\sum_{k=1}^{n} \Delta x \cdot f(a+(k-1)\Delta x)$ $\qquad \Delta x = \frac{b-a}{n}$

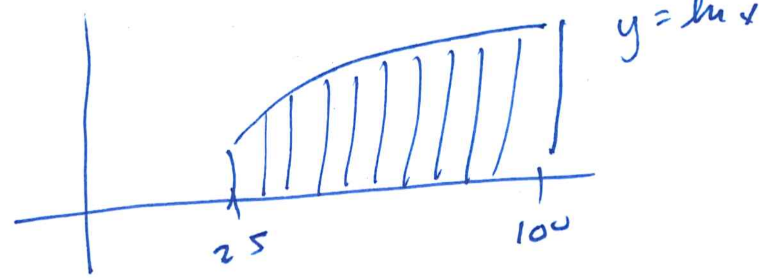
Right: $\sum_{k=1}^{n} \Delta x \cdot f(a+k\Delta x)$ $\qquad \Delta x = \frac{b-a}{n}$

Midpoint: $\sum_{k=1}^{n} \Delta x \cdot f(a+(k-\frac{1}{2})\Delta x)$

(ex) $f(x) = \ln x$

$[25, 100]$

$n = 10$ rectangles



$\Delta x = \frac{100-25}{10} = \frac{75}{10} = 7.5$

$a = 25$

Left RS:

$$\sum_{k=1}^{10} \Delta x \cdot f(a+(k-1)\Delta x)$$

$$= \sum_{k=1}^{10} (7.5) \ln(25+(k-1)\cdot 7.5)$$ some number

Right RS: $\sum_{k=1}^{10} (7.5) \cdot \ln(25+k\cdot 7.5)$ some number

Midpt RS: $\sum_{k=1}^{10} (7.5) \ln(25+(k-\frac{1}{2})7.5)$ some number

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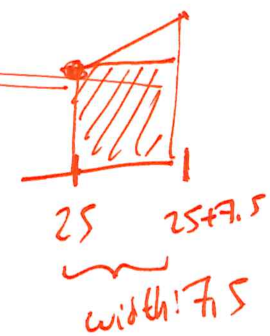
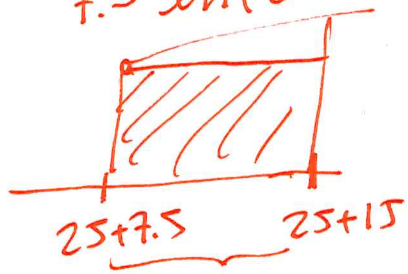
Left RS first terms: $(k=1)$ $7.5 \ln(25+0) = 7.5 \ln 25$



$(k=2)$ $7.5 \ln(25+7.5)$

area of 2nd ▯

etc....



Low-Degree Powers of k work nicely with Σ

p. 340 text

- $\displaystyle\sum_{k=1}^{n} c = \underbrace{c + c + c + \cdots + c}_{n \text{ times}} = cn$

  constant

  $k^0 = 1$

- $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$

  $\displaystyle = \frac{n(n+1)}{2}$

- $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

- $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

$1 + 2 + 3 + \cdots + 49 + 50 + 51 + 52 + \cdots + 97 + 98 + 99 + \underset{\overset{\shortparallel}{n}}{100}$

101 101 101 101 101

Sum: $101 \cdot 50 = (n+1)\left(\frac{n}{2}\right)$

ex Evaluate Rieman Sum (right)

$f(x) = x^2 + x$

$[1, 6]$

$n = 100$ rectangles

$\Delta x = \frac{6-1}{100} = \frac{5}{100} = \frac{1}{20}$

(width)

heigt: $f(x_k) = f(a + k\Delta x) = f\left(1 + k\cdot\frac{1}{20}\right)$

↑ right endpt

Sum: $\sum_{k=1}^{100} \underbrace{\frac{1}{20}}_{\substack{\Delta x \\ \text{width}}} \cdot \underbrace{f\Big(\overbrace{1+\frac{k}{20}}^{x}\Big)}_{\substack{\text{height} \\ @ \\ \text{right} \\ \text{endpt}}} = \sum_{k=1}^{100} \frac{1}{20}\cdot\left[\left(1+\frac{k}{20}\right)^2 + \left(1+\frac{k}{20}\right)\right]$

$= \frac{1}{20}\sum_{k=1}^{100}\left(1 + \frac{2k}{20} + \frac{k^2}{20^2} + 1 + \frac{k}{20}\right) = \frac{1}{20}\sum_{k=1}^{100}\left(2 + \frac{3}{20}k + \frac{1}{20^2}k^2\right)$

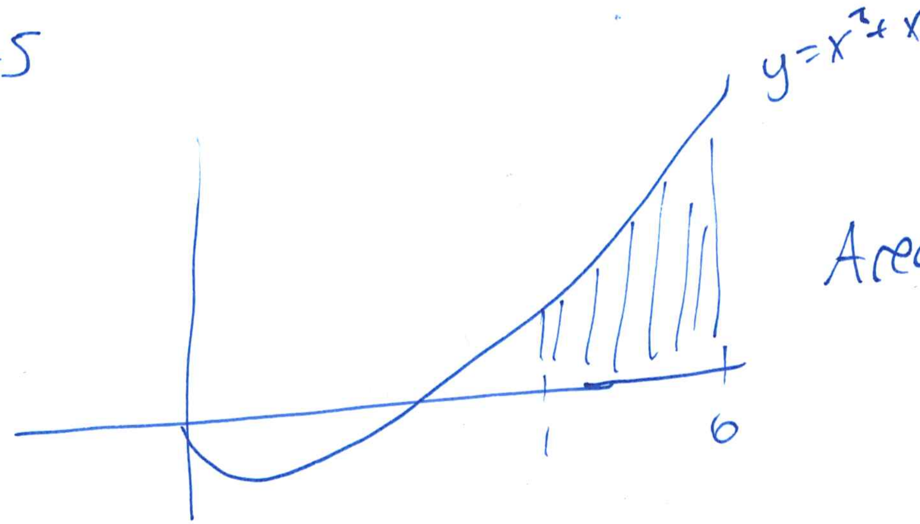
$$= \frac{1}{20}\left[\sum_{k=1}^{100} 2 + \sum_{k=1}^{100} \frac{3}{20}k + \sum_{k=1}^{100} \frac{1}{20^2}k^2\right]$$

$$= \frac{1}{20}\left[200 + \frac{3}{20}\sum_{k=1}^{100} k + \frac{1}{20^2}\sum_{k=1}^{100} k^2\right]$$

use list!

$$= \frac{1}{20}\left[200 + \frac{3}{20}\left(\frac{101\cdot 100}{2}\right) + \frac{1}{20^2}\left(\frac{100\cdot 101\cdot 201}{6}\right)\right.$$

$$= 90.16875$$



Area ≈ 90.16875