(ex) Ellipse: $x^2 - 2xy + 5y^2 = 1$

What are max/min values of x on ellipse?

want to max/minimize: $f(x,y) = x$

Constraint: $g(x,y) = x^2 - 2xy + 5y^2 = 1$

$f_x = 1 \qquad g_x = 2x - 2y$

$f_y = 0 \qquad g_y = -2x + 10y$

Solve:

$$\begin{cases} \cdot \quad 1 = \lambda(2x - 2y) \rightarrow \cancel{2x - 2y = 0} \text{ OR } \lambda = \frac{1}{2x-2y} \\ \cdot \quad 0 = \lambda(-2x + 10y) \rightarrow -2x + 10y = 0 \text{ OR } \lambda = 0 \\ \cdot \quad x^2 - 2xy + 5y^2 = 1 \end{cases}$$

If $2x - 2y = 0$: 1^{ST} Eq, LHS $= 1$, so never happens

If $-2x + 10y = 0$: $2x = 10y$, so $x = 5y$]

3^{RD} Eq: $(5y)^2 - 2(5y)y + 5y^2 = 1$

$25y^2 - 10y^2 + 5y^2 = 1$

$20y^2 = 1$

$y^2 = \frac{1}{20}$

$y = \pm\sqrt{\frac{1}{20}} = \frac{\pm 1}{2\sqrt{5}}$

Points to consider:

$\left(\frac{-5}{2\sqrt{5}}, \frac{-1}{2\sqrt{5}}\right) = \left(\frac{-\sqrt{5}}{2}, \frac{-1}{2\sqrt{5}}\right)$

$\left(\frac{5}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}\right) = \left(\frac{\sqrt{5}}{2}, \frac{1}{2\sqrt{5}}\right)$

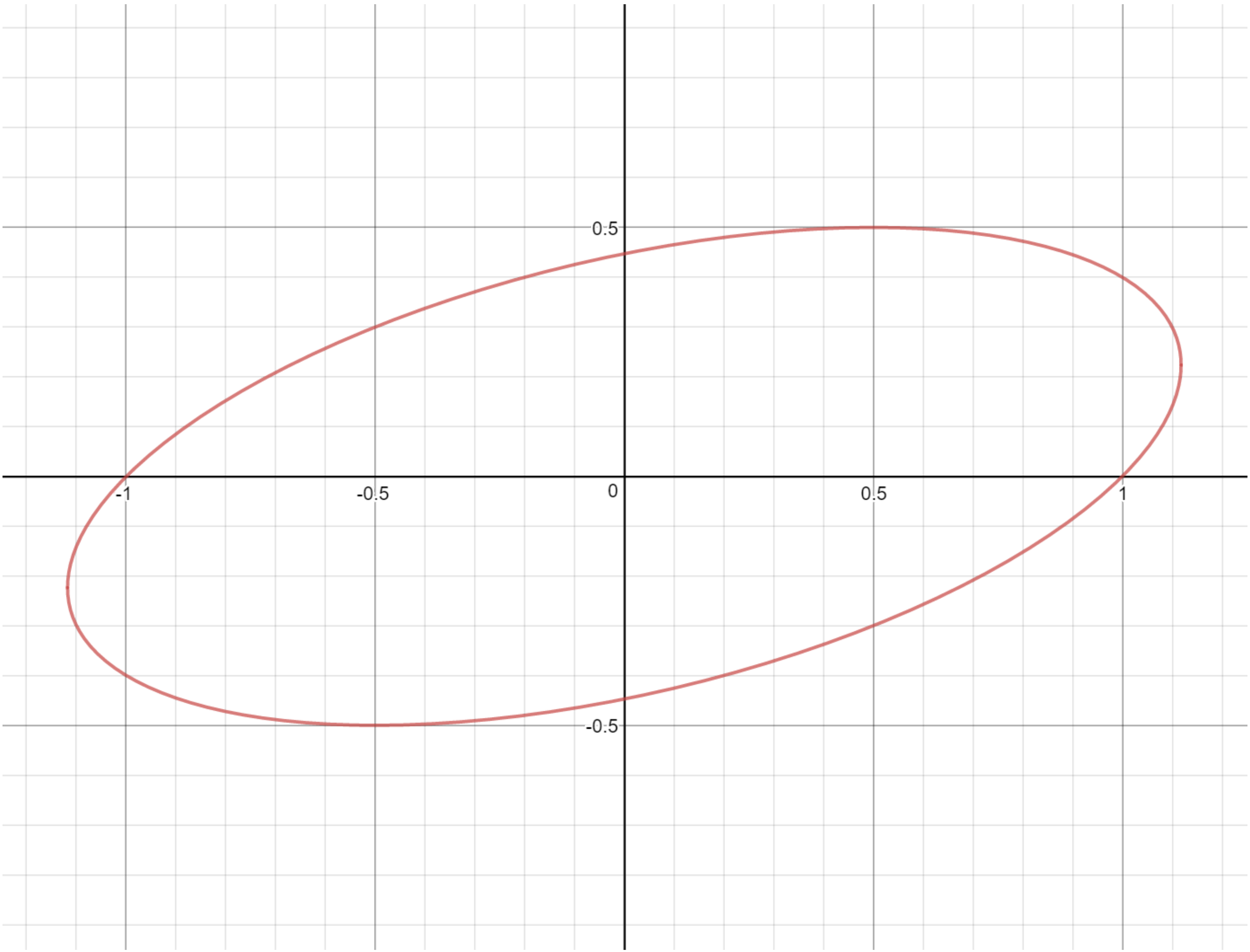
If $\lambda = 0$ and $\lambda = \frac{1}{2x-2y}$

then $0 = \frac{1}{2x-2y}$ never happens

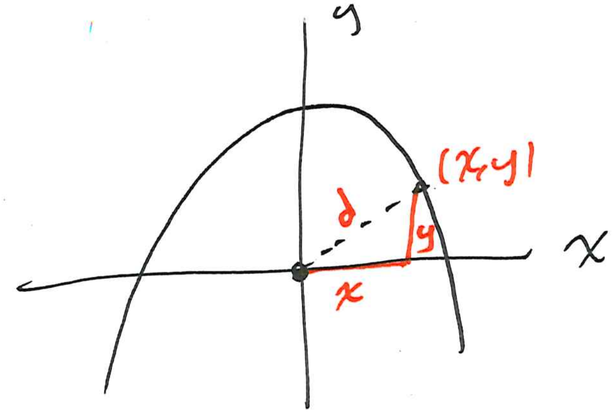
Points to consider:

$\left(\frac{\sqrt{5}}{2}, \frac{1}{2\sqrt{5}}\right)$ ← MAX x-value on ellipse

$\left(\frac{-\sqrt{5}}{2}, \frac{-1}{2\sqrt{5}}\right)$ ← MIN x-value on ellipse


0.5
-1
-0.5
0
0.5
1
-0.5

(ex) Find the point on $y = 1.5 - x^2$ closest to origin.



Want to minimize:
distance from pt to origin

$$d = \sqrt{x^2 + y^2}$$

$f(x,y) = x^2 + y^2$ } easier to work with

$f_x = 2x$
$f_y = 2y$

Constraint: $y = 1.5 - x^2$

$g(x,y) = y + x^2 = 1.5$

$g_x = 2x$
$g_y = 1$

Solve

$$\begin{cases} 2x = \lambda \cdot 2x \\ 2y = \lambda \cdot 1 \\ y + x^2 = 1.5 \end{cases}$$