Quiz 2:

Tuesday, Jan 24

12.2 – 12.9

Similar to suggested problems in book

(modified for time)

Quiz 1:

Hopefully in MLC after 4 PM

Worksheet to practice optimization is on the course website

Method of Lagrange Multipliers

Let the objective function f and the constraint function $g(x,y) = c$ be differentiable, with
constant

$g_x(x,y)$ and $g_y(x,y)$ not both always zero.

To locate the maximum and minimum values of f subject to the constraint $g(x,y) = c$:

① Find the values of x, y, and λ that satisfy the equations:

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = c \end{cases}$$

② Among the values (x,y) from Step 1, select the largest & smallest corresponding function values. These are the maximum & minimum values of f subject to the constraint.

(ex) The height of a roller coaster at (x, y) is

$$f(x,y) = xy + 14$$

But it only exists when $x^2+y^2=18$

What are highest & lowest points?

Objective (want to maximize / minimize)

$$f(x,y) = xy + 14$$

Constraint: $g(x,y) = x^2+y^2 = 18$

$f_x = y$ $\qquad g_x = 2x$

$f_y = x$ $\qquad g_y = 2y$

Solve:

$$\begin{cases} y = \lambda \cdot \underbrace{2x} \\ x = \lambda \cdot \underbrace{2y} \\ x^2+y^2 = 18 \end{cases}$$

$\cancel{2x=0}$ OR $\lambda = \frac{y}{2x}$

$\cancel{2y=0}$ OR $\lambda = \frac{x}{2y}$

If $2x=0$: Then $x=0$

1st Equation: $y=0$

Point $(0,0)$ NOT in constraint

No point to consider.

If $2y=0$: Then $y=0$

2nd Equation: $x=0$

$(0,0)$ not in constraint

No point to consider

So: $\frac{y}{2x} = \lambda = \frac{x}{2y}$, so $2y^2 = 2x^2$, so $x^2 = y^2$

3rd Equation: $x^2+x^2=18$

$2x^2=18$

$x^2=9$

$x=\pm 3$

Since $y^2=x^2$

$y^2=9$

$y=\pm 3$

Four points to consider: $(\pm 3, \pm 3)$

$f(x,y) = xy + 14$

$f(3,3) = f(-3,-3) = 9 + 14 = \boxed{23}$ highest pt (abs max)

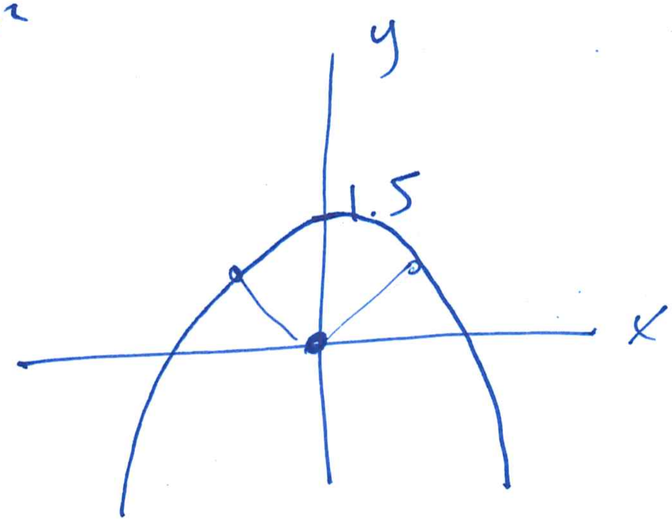
$f(3,-3) = f(-3,3) = -9 + 14 = \boxed{5}$ lowest pt (abs min)

along constraint

ex Find the point(s) on the parabola

$$y = 1.5 - x^2$$

closest to origin.



Want to minimize: distance to (0,0)

For any point (x,y):

$$D = \sqrt{x^2+y^2}$$

Easier:

$f(x,y) = x^2+y^2$ objective function

Only care about (x,y) if on parabola: $y = 1.5 - x^2$

$$y + x^2 = 1.5$$

Constraint:

$$g(x,y) = y + x^2 = 1.5$$

$f_x = 2x$ $\quad g_x = 2x$

$f_y = 2y$ $\quad g_y = 1$

Solve:

$$\begin{cases} 2x = \lambda \cdot \underbrace{2x} \\ 2y = \lambda \cdot 1 \\ y + x^2 = 1.5 \end{cases}$$

$\to 2x = 0$ OR $\lambda = \frac{2x}{2x} = 1$

$\to \lambda = 2y$

If $2x = 0$: $(x=0)$

3rd Equation: $y + 0 = 1.5$

$y = 1.5$

Point to consider: $(0, 1.5)$

Otherwise: (1ST) $\lambda = 1$ and (2nd) $\lambda = 2y$

$1 = 2y$, so $y = 1/2$

3rd: $\frac{1}{2} + x^2 = 1.5$

$x^2 = 1$

$x = \pm 1$

Points to consider: $(1, \frac{1}{2})$

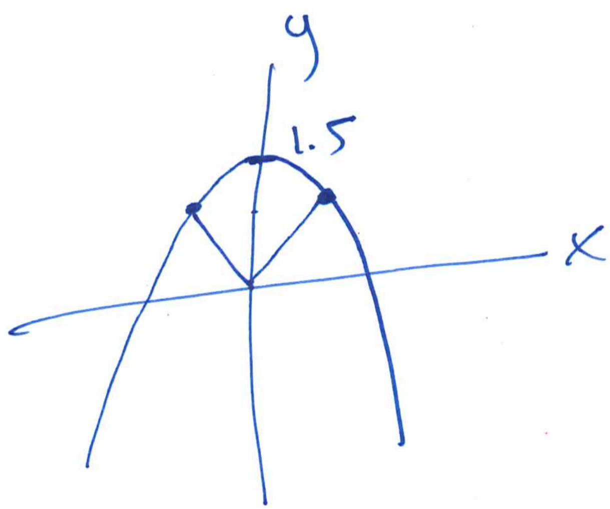
$(-1, \frac{1}{2})$

$f(x,y) = x^2 + y^2$

- $f(0, 1.5) = 0^2 + \left(\frac{3}{2}\right)^2 = \frac{9}{4}$
- $f(1, \frac{1}{2}) = 1^2 + \left(\frac{1}{2}\right)^2 = 1 + \frac{1}{4} = \frac{5}{4}$
- $f(-1, \frac{1}{2}) = (-1)^2 + \left(\frac{1}{2}\right) = \frac{5}{4}$

$\left.\begin{matrix} \frac{5}{4} \\ \frac{5}{4} \end{matrix}\right\}$ smaller

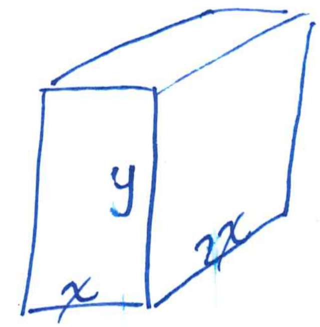
Closest points: $(1, \frac{1}{2})$ & $(-1, \frac{1}{2})$



ex A rectangular box has volume 72 cu ft.

Its width is twice its length.

What is the minimum possible surface area, and what dimensions achieve this?



Objective fn: surface area

surface area:

$$2(xy)+2(2xy)+2(2x^2)$$
$$= 2xy+4xy+4x^2$$
$$= 6xy+4x^2$$

$$f(x,y) = 6xy+4x^2$$

$f_x = 6y+8x$ $\quad g_x = 4xy$

$f_y = 6x$ $\quad g_y = 2x^2$

Constraint: volume

$$g(x,y) = 2x^2y = 72$$

Solve:

$$\begin{cases} 6y+8x = \lambda \cdot \underbrace{4xy} & \rightarrow \cancel{4yx=0} \quad \text{OR} \quad \lambda = \dfrac{6y+8x}{4xy} = \dfrac{3y+4x}{2xy} \\ 6x = \lambda \cdot 2x^2 & \rightarrow \cancel{2x^2=0} \quad \text{OR} \quad \lambda = \dfrac{6x}{2x^2} = \dfrac{3}{x} \\ 2x^2y = 72 & \end{cases}$$

By Eqn #3: $x \neq 0,\ y \neq 0$

So $4yx \neq 0$ and $2x^2 \neq 0$

So: $\underset{(1st)}{\dfrac{3y+4x}{2xy}} = \lambda = \underset{(2nd)}{\dfrac{3}{x}}$

$$3xy + 4x^2 = 6xy$$

$$4x^2 = 3xy$$

$$4x = 3y \qquad \text{Since } x \neq 0, \text{ cancel it}$$

$$\cancel{2x = 3y}$$

$$\cancel{x = \tfrac{3}{2}y}$$

$$y = \frac{4}{3}x$$

3rd Eqn: $2x^2\left(\frac{4}{3}x\right) = 72$

$$\frac{\cancel{8}}{3}x^3 = 72 = 9 \cdot \cancel{8}$$

$$x^3 = 27$$

$$x = 3 \rightarrow y = \frac{4}{3}(3)$$

$$y = 4$$

Point: $(3,4)$

1 option: There is no max surface area, so (3,4) must give min

Another option: Choose any other point. (on constraint)

ex: $2x^2y = 72$

$x = 1$
$y = 36$ (one of many possible choices)

Surface area: $f(1,36) = 6(1)(36) + 4(1)^2$
$= 6 \cdot 36 + 4 = 18 \cdot 12 + 4$ ← bigger

Compare: $f(3,4) = 6(3)(4) + 4(3)^2$
$= 6 \cdot 12 + 12 \cdot 3$
$= 12 \cdot 9$ ← NOT MAX

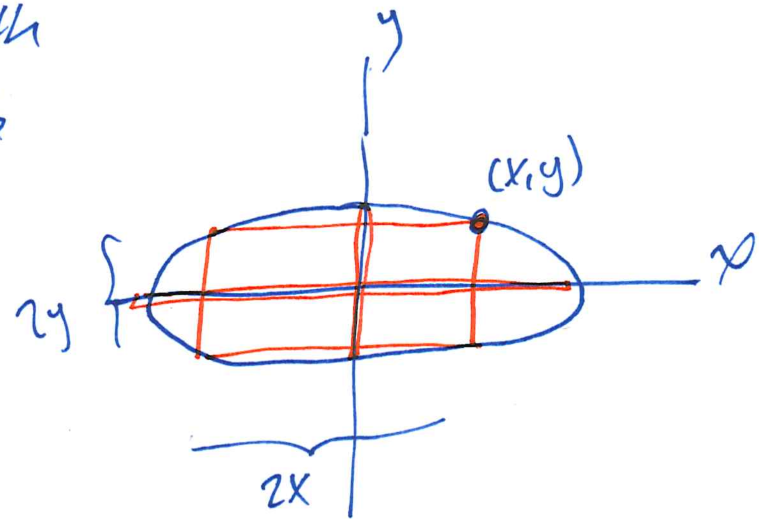
MIN surface area: $12 \cdot 9$ sq ft

Dimensions: $(3) \times (6) \times (4)$
$x \quad 2x \quad y$

Third option: Question asked for a min; this should be a min ex What is the maximum area of a rectangle, with sides parallel to coordinate axes, inscribed in the ellipse

$$4x^2+16y^2=16$$

?



Objective (maximize): Area

Area: $(2x)(2y)$ if (x,y) is the point in 1st quadrant where rect, ellipse touch

$f(x,y) = 4xy$ $(4|x||y|)$
$(x,y>0)$

Constraint: $g(x,y) = 4x^2+16y^2 = 16$

$f_x = 4y$ $g_x = 8x$

$f_y = 4x$ $g_y = 32y$

Solve :

$$\begin{cases} 4y = \lambda \cdot \underbrace{8x} \longrightarrow \cancel{8x=0} \quad \text{OR} \quad \lambda = \frac{4y}{8x} = \frac{y}{2x} \\ 4x = \lambda \cdot \underbrace{32y} \longrightarrow \cancel{32y=0} \quad \text{OR} \quad \lambda = \frac{4x}{32y} = \frac{x}{8y} \\ 4x^2+16y^2=16 \end{cases}$$

If $8x=0$, then $x=0$, 1^{ST} Eq : $y=0$ BUT: $(0,0)$ not in constraint

If $32y=0$, then $y=0$, 2^{ND} Eq : $x=0$ BUT: 3^{rd} Eq: not in constraint

Last case : $\underset{(1st)}{\frac{y}{2x}} = \lambda = \underset{(2nd)}{\frac{x}{8y}}$

$$8y^2 = 2x^2$$

$$4y^2 = x^2$$

$$16y^2 = 4x^2$$

3^{rd} Eq: $4x^2 + (4x^2) = 16$

$$8x^2 = 16$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

$$4y^2 = 2$$

$$y^2 = \frac{1}{2}$$

$$y = \pm\frac{1}{\sqrt{2}}$$

Points to Consider:

$$\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}\right)$$

$f(x, y) = 4xy$

$f(\sqrt{2}, \frac{1}{\sqrt{2}}) = 4(\sqrt{2})(\frac{1}{\sqrt{2}}) = 4$

$\cancel{f(-\sqrt{2}, \frac{-1}{\sqrt{2}}) = 4}$

we chose x, y in 1st quadrant so $x, y \geq 0$

4: max area.