Inserted Section:

Partial Derivatives

using

Implicit Differentiation

z: "function"

$z = f(x, y)$

$$\underbrace{z \sin x} + \underbrace{y \sin z} = 0$$

differentiate with respect to x:

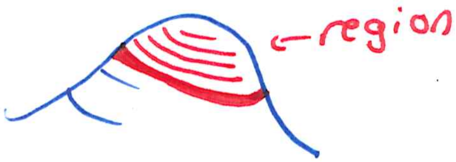
y - constant

z - function of x

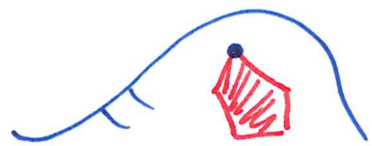
$$\underbrace{z \cdot \cos x + \sin x \cdot z_x} + \underbrace{y \cdot \cos z \cdot z_x} = 0$$

Absolute Maxima & Minima

(over a bounded region)



Max @ critical pt
(this example)



MAX @ bounday ← like endpoints
(this example)

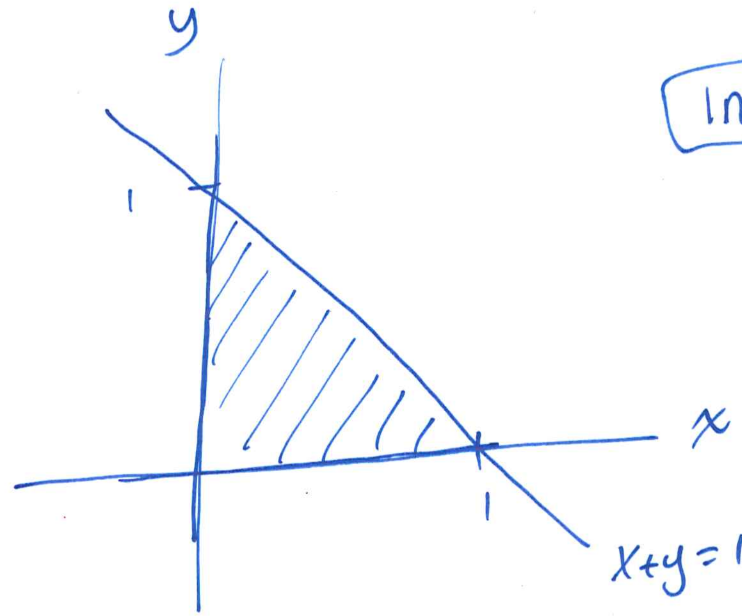
ex

$$f(x,y) = \frac{x+1}{x+y+1}$$

Find absolute max + min values of $f(x,y)$

when:

- $x+y \leq 1$
- $x \geq 0$
- $y \geq 0$



Inputs

Plan:

- Find CPs ← don't need 2nd deriv. test
- Find extrema along bounday
- Compare

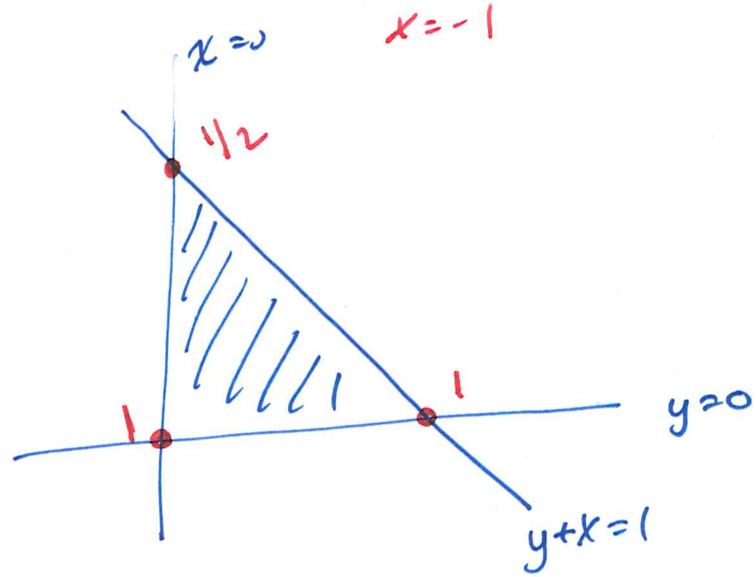
$\boxed{\text{Find CPs}}$ $\quad f(x,y) = \dfrac{x+1}{x+y+1}$

$$f_x = \frac{(x+y+1)(1) - (x+1)(1)}{(x+y+1)^2} = \frac{y}{(x+y+1)^2} = 0 \longrightarrow y = 0$$

$$f_y = \frac{(x+y+1)(0) - (x+1)(1)}{(x+y+1)^2} = \frac{-(x+1)}{(x+y+1)^2} = 0 \longrightarrow x = -1$$

$-(x+1) = 0$
$x+1 = 0$
$x = -1$

CP: $\left.\begin{array}{l} x=-1 \\ y=0 \end{array}\right\}$ not in region — ignore



$\boxed{\text{Check Boundary}}$

$\boxed{y+x=1}$ $\quad 0 \le x \le 1$

In that case:

$$f(x,y) = \frac{x+1}{x+y+1} = \frac{x+1}{2}$$

$$f(0,1) = \frac{0+1}{2} = 1/2$$

$$f(1,0) = \frac{1+1}{2} = 1$$

$\boxed{x=0}$ $\quad 0 \le y \le 1$

$$f(0,y) = \frac{0+1}{0+y+1} = \frac{1}{y+1}$$

$$f(0,1) = \frac{1}{1+1} = \frac{1}{2}$$

$$f(0,0) = \frac{1}{0+1} = 1$$

$\boxed{y=0}$ $\quad 0 \le x \le 1$

$$f(x,0) = \frac{x+1}{x+0+1} = \frac{x+1}{x+1} = 1$$

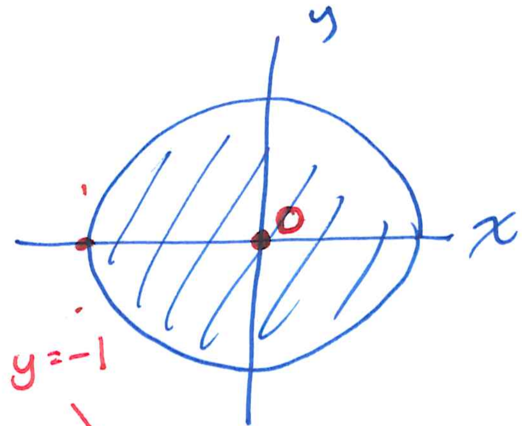
MAX: $f(x,y) = 1$ when $y = 0$

MIN: $f(x,y) = 1/2$ at $(0,1)$

ex $f(x,y) = x^2 + x^2y + y^2$

Find absolute extrema when $x^2+y^2 \le 1$

max/min



Critical Points

$$\begin{cases} f_x = 2x + 2xy = 2x(1+y) = 0 \longrightarrow x=0 \text{ OR } y=-1 \\ f_y = x^2 + 2y = 0 \longrightarrow \end{cases}$$

If $x=0$:

$2y=0$

$y=0$

If $y=-1$:

$x^2 - 2 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

$f(0,0) = 0$

CP: $(0,0)$

$\cancel{(-\sqrt{2},-1)}$

$\cancel{(\sqrt{2},-1)}$

not in region

$\boxed{\text{Boundary}}$ If (x,y) is a point of boundary (not inside):

$x^2+y^2=1 \longrightarrow x^2=1-y^2$

Then: $f(x,y) = x^2+x^2y+y^2$

$= (1-\cancel{y^2}) + (1-y^2)y + \cancel{y^2}$

$= 1+y-y^3 \leftarrow$ where is this MAX / MIN

$-1 \le y \le 1$

Subproblem: $g(y) = -y^3+y+1$

$-1 \le y \le 1$

Find extrema

$g'(y) = -3y^2+1 = 0$

$1 = 3y^2$

$\frac{1}{3} = y^2 \qquad y = \pm\frac{1}{\sqrt{3}}$

$g(1) = -1+1+1 = 1$

$g(-1) = -(-1)-1+1 = 1-1+1 = 1$

$g\left(\frac{1}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}^3} + \frac{1}{\sqrt{3}}+1 = \frac{1}{\sqrt{3}}\left(1-\frac{1}{3}\right)+1 = \boxed{\frac{2}{3\sqrt{3}}+1}$ MAX: when $y = \frac{1}{\sqrt{3}}$

$g\left(\frac{-1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}^3} - \frac{1}{\sqrt{3}}+1 = \frac{1}{\sqrt{3}}\left(\frac{1}{3}-1\right)+1 = \boxed{\frac{-2}{3\sqrt{3}}+1}$ MIN: when $y = \frac{-1}{\sqrt{3}}$

Compare:

interior $\{ \; f(0,0) = 0 \; \Big]$ ABSOLUTE MIN

boundary $\left\{ \begin{array}{l} f(\quad, \frac{1}{\sqrt{3}}) = \frac{2}{3\sqrt{3}} + 1 \\ f(\quad, \frac{-1}{\sqrt{3}}) = 1 - \frac{2}{3\sqrt{3}} \end{array} \right.$ $\Big]$ ABSOLUTE MAX

$y = \frac{1}{\sqrt{3}} \rightarrow y^2 = \frac{1}{3}$

$x^2 = 1 - y^2 = 1 - \frac{1}{3} = 2/3$

$\boxed{x = \pm\sqrt{\frac{2}{3}}}$

$\boxed{y = \frac{1}{\sqrt{3}}}$

Which is the smallest value?

$3\sqrt{3} > 3$

So: $\frac{1}{3\sqrt{3}} < \frac{1}{3}$

So: $\frac{2}{3\sqrt{3}} < \frac{2}{3}$

So: $\frac{-2}{3\sqrt{3}} > \frac{-2}{3}$

So: $1 - \frac{2}{3\sqrt{3}} > 1 - \frac{2}{3} = 1/3 > 0$

0 is the smallest value.

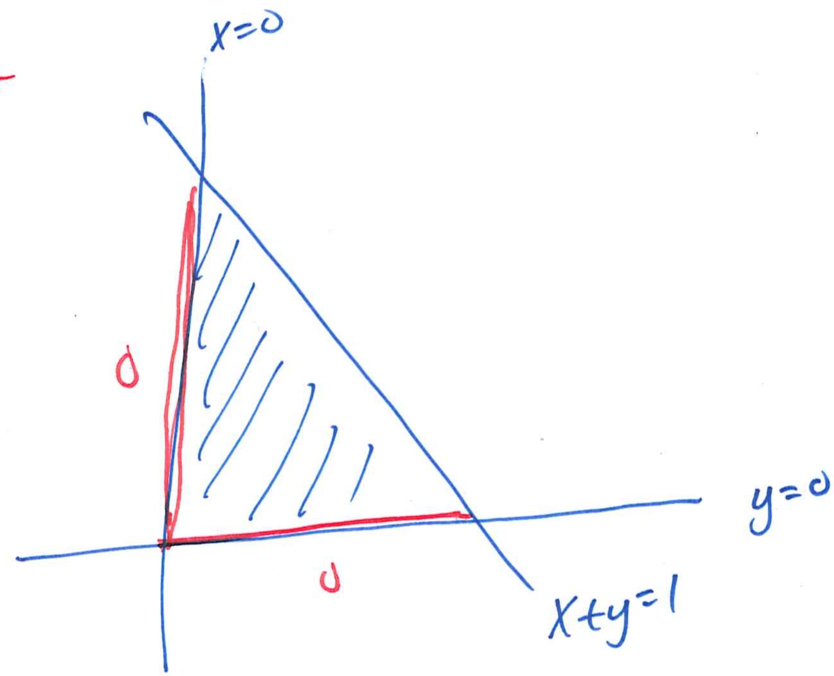
ex

$$f(x,y) = (xy)e^{-x-y}$$

Region:

$$R = \underbrace{\{}_{\text{collection}} \underbrace{(x,y)}_{\text{of all of these}} \underbrace{:}_{\text{such that}} \underbrace{x \geq 0,\ y \geq 0,\ x+y \leq 1}_{\text{these true}} \}$$

Find abs max/min
of $f(x,y)$ over R



Plan:
- Find CPs
- Find extrema on boundary
- Compare

$\boxed{\text{CPs}}$ $f(x,y) = (xy)\,e^{-x-y}$

$f_x = (xy)\cdot e^{-x-y}(-1) + e^{-x-y}\cdot y$

$= ye^{-x-y}(1-x) = 0 \longrightarrow \boxed{y=0 \quad \underline{\text{OR}} \quad x=1}$

AND

$f_y = xe^{-x-y}(1-y) = 0 \longrightarrow \boxed{x=0 \quad \underline{\text{OR}} \quad y=1}$

CP: $\cancel{(1,1)}$ not in R

$\boxed{(0,0)}$ $\boxed{f(0,0)=0}$

$$f(x,y) = (xy)e^{-x-y}$$

Boundaries

$\boxed{x=0}$ $\quad f(x,y) = 0$

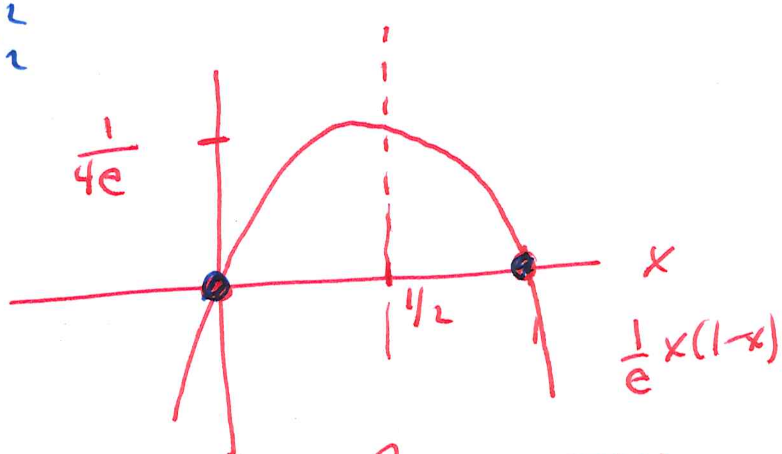
$\boxed{y=0}$

$\boxed{x+y=1}$ $\quad f(x,y) = (xy)e^{-(x+y)} = xye^{-1} = \dfrac{xy}{e} = \underbrace{\dfrac{1}{e}x(1-x)}_{\text{parabola } \cap}$

$y = 1-x \rightarrow$ if $x = 1/2$ then $y = 1/2$

$0 \le x \le 1$

intercepts: $x=0$, $x=1$



MIN: when $x=0$ or $x=1$, $fcn = 0$

MAX: when $\boxed{x = 1/2}$

fcn: $\dfrac{1}{e}\left(\dfrac{1}{2}\right)\left(1-\dfrac{1}{2}\right)$

$= \dfrac{1}{e}\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)$

$= \dfrac{1}{4e}$

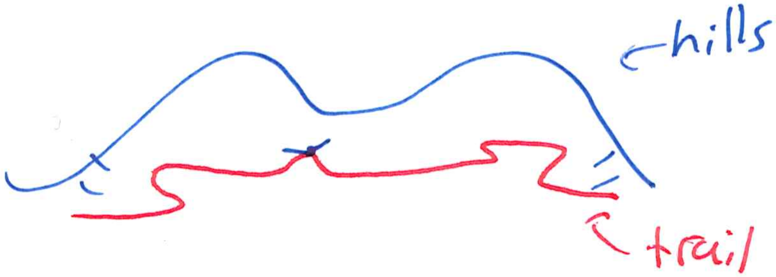
COMPARE

CP: $f(0,0) = 0$

Bdry: If $x=0$ or $y=0$, $f(x,y) = 0$ $\Big\}$ 0 : MIN

If $x = 1/2$ $y = 1/2$, $f(1/2, 1/2) = \dfrac{1}{4e}$ $\Big\}$ MAX

Ch 12.9 Method of Lagrange Multipliers (Constrained optimization)



find highest pt of trail (boundary)

• "We can use this method for abs max/min over a bounded region when boundary is hard to plug in."

Method of Lagrange Multipliers

Suppose we want to find abs max/min of a function

$$f(x,y)$$ } "objective function"

subject to the constraint

$$g(x,y) = \underset{\substack{\uparrow \\ \text{constant}}}{c}$$

These will only occur at points (a,b) such that, for some constant λ, all three equations below are true:

$$\begin{cases} ① \quad f_x(a,b) = \lambda g_x(a,b) \\ ② \quad f_y(a,b) = \lambda g_y(a,b) \\ ③ \quad g(x,y) = c \end{cases}$$

How do we solve these 3 equations?

(1) $g_x(a,b) = 0$ OR $\lambda = \dfrac{f_x(a,b)}{g_x(a,b)}$

(2) $g_y(a,b) = 0$ OR $\lambda = \dfrac{f_y(a,b)}{g_y(a,b)}$

MAYBE: $g_x(a,b) = 0$ and $f_x(a,b) = 0$

MAYBE: $g_y(a,b) = 0$ and $f_y(a,b) = 0$

MAYBE: $\dfrac{f_x(a,b)}{g_x(a,b)} = \dfrac{f_y(a,b)}{g_y(a,b)}$

ALSO: $g(x,y) = c$