Second Derivative Test
(Theorem 12.14, p941)

Suppose the second partial derivatives of f are continuous; and at a point (a,b),

$$f_x(a,b) = f_y(a,b) = 0$$ ← critical point

discriminant

Let $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2$.

① If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then (a,b) gives a local MAX of f

② If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then (a,b) gives a local MIN of f

③ If $D(a,b) < 0$, then f has a saddle point at (a,b)

④ If $D(a,b) = 0$, test is inconclusive.

(ex) $f(x,y) = x^2 - 2x - y^2 - 4y - 4$

Find all local max/min, Saddle Pts

1. Find all CPs

$$\begin{cases} f_x = 2x - 2 \\ f_y = -2y - 4 \end{cases} \qquad \begin{cases} 0 = 2x - 2 \\ 0 = -2y - 4 \end{cases} \qquad CP: (1, -2)$$

2. Find 2nd Derivatives

$$f_{xx} = 2 \qquad f_{xy} = 0 \qquad f_{yy} = -2$$

$$D = f_{xx} f_{yy} - f_{xy}^2 = (2)(-2) - 0 = -4 < 0$$

3. Use test: $(1, -2)$ is a saddle pt

(NO local max or min)

(ex) $f(x,y) = x^2 - 6x + \frac{1}{4}y^4 - 8y$

Find all local max/min, saddle pts

$$\begin{cases} f_x = 2x - 6 \\ f_y = y^3 - 8 \end{cases} \qquad \begin{cases} 0 = 2x - 6 \\ 0 = y^3 - 8 \end{cases} \qquad \begin{cases} 2x = 6 \\ y^3 = 8 \end{cases}$$

$$\begin{cases} x = 3 \\ y = 2 \end{cases} \qquad \boxed{CP: (3, 2)}$$

$(-2)^3 =$

$(-2)(-2)(-2) =$

$4(-2) = -8$

$f_{xx} = 2 \qquad f_{xy} = 0 \qquad f_{yy} = 3y^2$

$$D(3,2) = f_{xx}(3,2) \cdot f_{yy}(3,2) - [f_{xy}(3,2)]^2$$
$$= (2)(3 \cdot 2^2) - 0 > 0$$

$f_{xx} > 0$

$\boxed{\text{at } (3,2), \ f \text{ has a local min}}$

no local max,
no saddle point

ex) $f(x,y) = x^2 + xy^2 - 2x + 1$

find all local max/min, saddle pts

$$\begin{cases} f_x = 2x + y^2 - 2 \\ f_y = 2xy \end{cases}$$

$$\begin{cases} 0 = 2x + y^2 - 2 \\ 0 = 2xy \end{cases}$$

$$\begin{cases} 0 = y^2 - 2 \\ x = 0 \end{cases} \quad \text{or} \quad \begin{cases} 0 = 2x - 2 \\ y = 0 \end{cases}$$

$$= \begin{cases} y = \pm\sqrt{2} \\ x = 0 \end{cases} \quad \text{or} \quad \begin{cases} x = 1 \\ y = 0 \end{cases}$$

CPs:

$(0, -\sqrt{2})$

$(0, \sqrt{2})$

$(1, 0)$

$f_{xx} = 2$ $\quad$ $f_{xy} = 2y$

$f_{yy} = 2x$

$$D(x,y) = (2)(2x) - (2y)^2$$
$$= 4x - 4y^2$$
$$= 4(x - y^2)$$

If $(x,y) = (0,-\sqrt{2})$, then $D(0,-\sqrt{2}) = 4(0-2) < 0$

If $(x,y) = (0,\sqrt{2})$, then $D(0,\sqrt{2}) = 4(0-2) < 0$

SADDLE PT @ $(0,-\sqrt{2})$ $(0,\sqrt{2})$

If $(x,y) = (1,0)$ then $D(1,0) = 4(1-0) > 0$

$f_{xx} = 2 > 0$

Local min @ $(1,0)$

ex Find all CPs of:

$$f(x,y) = x^2+xy+y^2+6x+9$$

$$\begin{cases} f_x = 2x+y+6 \\ f_y = x+2y \end{cases}$$

$$\begin{cases} 0 = 2x+y+6 \\ 0 = x+2y \end{cases}$$

$$\begin{cases} 0 = 2x+y+6 \\ x = -2y \end{cases}$$

$$\begin{cases} 0 = 2(-2y)+y+6 \\ x = -2y \end{cases}$$

$$\begin{cases} 0 = -4y+y+6 \\ x = -2y \end{cases}$$

$$\begin{cases} 0 = -3y+6 \\ x = -2y \end{cases}$$

$$\begin{cases} y = 2 \\ x = -2y \end{cases}$$

$$\begin{cases} y = 2 \\ x = -4 \end{cases}$$

$$\boxed{CP: (-4, 2)}$$

(ex) Find all CPs of

$$f(x,y) = x^2 + y^3 + xy + 5x$$

$$\begin{cases} f_x = 2x + y + 5 \\ f_y = 3y^2 + x \end{cases}$$

$$\begin{cases} 0 = 2x + y + 5 \\ 0 = 3y^2 + x \end{cases}$$

$$\begin{cases} 0 = 2(-3y^2) + y + 5 \\ x = -3y^2 \end{cases}$$

$$\begin{cases} 0 = -6y^2 + y + 5 \\ x = -3y^2 \end{cases}$$

TOP EQUATION:

$$y = \frac{-1 \pm \sqrt{1 - 4(-6)(5)}}{-12}$$

$$= \frac{-1 \pm \sqrt{1 + 120}}{-12}$$

$$= \frac{-1 \pm \sqrt{121}}{-12} = \frac{-1 \pm 11}{-12}$$

$$\nearrow \frac{-1+11}{-12} = \frac{10}{-12} = -5/6$$

$$\searrow \frac{-1-11}{-12} = \frac{-12}{-12} = 1$$

CASE 1: $y = -5/6$

$$x = -3(-5/6)^2 = -3\left(\frac{25}{36}\right)$$

CASE 2: $y = 1$

$$x = -3(1) = -3$$

CPs: $\left(\frac{-3 \cdot 25}{36}, \frac{-5}{6}\right)$ or $(-3, 1)$

Using inspection to classify CPs

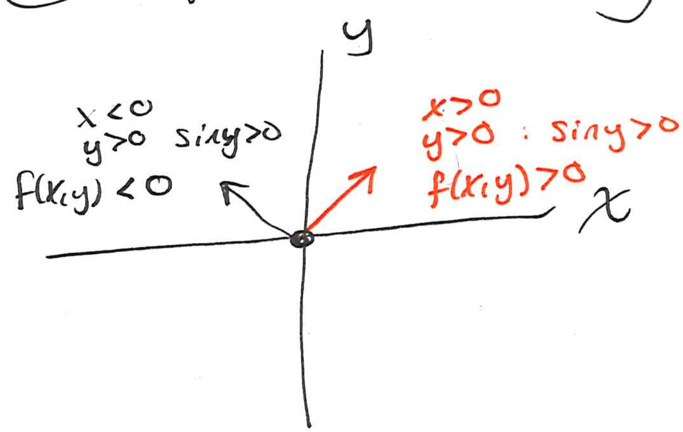
(ex) $f(x,y) = \cos(xy)$

Point: $(0,0)$

$f(0,0) = \cos(0 \cdot 0) = 1$

Since cosine ≤ 1 for any input,
$(0,0)$ gives local max

(ex) $f(x,y) = x \sin y$



Point: $(0,0)$

$f(0,0) = 0 \cdot \sin 0 = 0$

So: $(0,0)$ saddle point
(not max or min)